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The r-modes of rotating fluids

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Abstract. An analysis of the toroidal modes of a rotating fluid, by means of the differential equations of motion, is not readily tractable. A matrix representation of the equations on a suitable basis, however, simplifies the problem considerably and reveals many of its intricacies. Let Ω be the angular velocity of the star and (ℓ, m) be the two integers that specify a spherical harmonic function. One readily finds the followings: 1) Because of the axial symmetry of equations of motion, all modes, including the toroidal ones, are designated by a definite azimuthal number m. 2) The analysis of equations of motion in the lowest order of Ω shows that Coriolis forces turn the neutral toroidal motions of (ℓ, m) designation of the non-rotating fluid into a sequence of oscillatory modes with frequencies $2m\Omega/\ell(\ell+1)$. This much is common knowledge. One can say more, however. a) Under the Coriolis forces, the eigendisplacement vectors remain purely toroidal and carry the identification (ℓ, m). They remain decoupled from other toroidal or poloidal motions belonging to different ℓ's. b) The eigenfrequencies quoted above are still degenerate, as they carry no reference to a radial wave number. As a result the eigendisplacement vectors, as far as their radial dependencies go, remain indeterminate. 3) The analysis of the equation of motion in the next higher order of Ω reveals that the forces arising from asphericity of the fluid and the square of the Coriolis terms (in some sense) remove the radial degeneracy. The eigenfrequencies now carry three identifications (s, ℓ, m), say, of which s is a radial eigennumber. The eigendisplacement vectors become well determined. They still remain zero order and purely toroidal motions with a single (ℓ, m) designation. 4) Two toroidal modes belonging to ℓ and $\ell \pm 2$ get coupled only at the Ω^2 order. 5) A toroidal and a poloidal mode belonging to ℓ and $\ell \pm 1$, respectively, get coupled but again at the Ω^2 order. Mass and mass-current multipole moments of the modes that are responsible for the gravitational radiation, and bulk and shear viscosities that tend to damp the modes, are worked out in much detail.

Key words. stars: neutron – stars: oscillations – stars: rotation

1. Introduction

Recent years have seen a surge of interest in the small oscillations of rotating fluid masses. The reason for the excitement is the advocation by relativists that in rapidly rotating neutron stars, the gravitational radiation drives the r-modes to become unstable, and while spinning down the star, may itself be amenable to detection (see the recent review by Andersson & Kokkotas 2000). Nevertheless, the oscillations of rotating objects is an old problem. In the past few decades it has been studied by many investigators and from various points of view. Complexity of the problem arises from the fact that a fluid can support three distinct types of motions, derived from, say, a scalar potential, from a toroidal vector potential and from a poloidal vector potential. These are the motions associated predominantly with the familiar p-, g- and toroidal- oscillations of the fluid. Each of these motions, in turn, can be given an expansion in terms of vector spherical harmonics. The modes of an actual star are a mixture of the three types mentioned above and of the different spherical harmonic components. Sorting out this mixture and classifying the modes into well-defined sequences has not been an easy task. Moreover, and more often than not, it has not been realized that g-modes of spherically asymmetric configurations are not apt for perturbation analysis as the low frequency tail of their spectrum is a fragile structure. It is driven by minute buoyancy forces and can be completely wiped out by almost any perturbing agent, such as Coriolis, asphericity and magnetic forces.

Here we show that a matrix representation of the equations of motions provides a set of algebraic equations that are much easier to cope with than their differential

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counterparts. In Sect. 2 we write down the equilibrium structure and the linearized equations of motion of a rotating star. In Sect. 3 we introduce the matrix representation of these equations. In Sects. 4 and 5 we sort out the toroidal-poloidal components and spherical harmonic constituents of the matrices. In Sects. 6 and 7 we give an ordering of the various components in powers of Ω^2 and sort out the equations of motion in various orders of magnitude. References and bibliographical notes relating to mode calculations in rotating stars are collected in Sect. 7.4. In Sect. 8 we discuss the numerical results. A rotating neutron star can be slowed down by gravitational radiation through the mass and mass-current multipole moments of the modes. The modes, in turn, can be damped out by bulk and shear viscosities present in the star. The time scales of relevant damping mechanisms are analyzed in Sect. 9. Calculations of matrix elements and presentation of appropriate basis sets are given in the appendices.

2. Review of rotating fluids

2.1. Equilibrium configuration

Let $\rho(r,\theta)$, $p(r,\theta)$ and $U(r,\theta)$ be the density, the pressure and the gravitational potential of a star rotating with the constant angular frequency Ω about the z-axis. The equilibrium condition is

$$\boldsymbol{\nabla} p + \rho \boldsymbol{\nabla}\left[U - \frac{1}{2}\Omega^2 r^2 \cos^2\theta\right] = 0. \tag{1}$$

For slow rotations, one obtains

$$\rho = \rho_0(r) + \Omega^2[\rho_{20}(r) + \rho_{22}(r) P_2(\cos\theta)], \tag{2a}$$

where $P_2(\cos\theta)$ is a Legendre polynomial. Similar expansions exists for p and U. For a barotropic structure, $p(\rho)$, one will have

$$p_0(\rho_0), \text{ and } p_{2i} = \frac{\mathrm{d}p_0}{\mathrm{d}\rho_0}\rho_{2i};\ i = 0, 2. \tag{2b}$$

Furthermore, Poisson's equation will give

$$\nabla^2(U_0, U_{20}, U_{22}) = 4\pi G(\rho_0, \rho_{20}, \rho_{22}). \tag{2c}$$

A thorough study of the structure of rotating polytropes was given as early as 1933 by Chandrasekhar. Further numerical values of ρ_{20} and ρ_{22} may be found in Chandrasekhar & Lebovitz (1962).

2.2. Linear perturbations

Let a mass element of the rotating fluid at position $\mathbf{x}$ be displaced by an amount $\boldsymbol{\xi}^s(\mathbf{x})\exp(i\omega^s t)$, where, for the moment, s is the collection of all indices that specify the displacement in question. This may include its spherical harmonic specifications, its radial node number, and/or its poloidal and toroidal nature. The Eulerian change in ρ, and U resulting from this displacement, will be

$$\delta^s\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}^s), \tag{3a}$$

$$\delta^s U(\mathbf{x}) = -G\int \delta^s\rho(\mathbf{x}') \mid \mathbf{x} - \mathbf{x}' \mid^{-1} \mathrm{d}^3x'. \tag{3b}$$

On the assumption that the displacement takes place adiabatically and the Lagrangian change in pressure is $-(\partial p/\partial\rho)_{\mathrm{ad}}\rho\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, one obtains

$$\begin{aligned}\delta^s p &= -(\partial p/\partial\rho)_{\mathrm{ad}}\rho\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^s - \boldsymbol{\nabla} p\cdot\boldsymbol{\xi}^s \\ &= (\mathrm{d}p/\mathrm{d}\rho)\ \delta^s\rho + [\mathrm{d}p/\mathrm{d}\rho - (\partial p/\partial\rho)_{\mathrm{ad}}]\rho\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^s,\end{aligned} \tag{3c}$$

where $\mathrm{d}p/\mathrm{d}\rho$ is the barotropic derivative of the equilibrium structure. The linearized Euler equation governing the evolution of $\boldsymbol{\xi}^s$ now becomes

$$\mathcal{W}\boldsymbol{\xi}^s + 2i\omega^s\rho\boldsymbol{\Omega}\times\boldsymbol{\xi}^s - {\omega^s}^2\rho\boldsymbol{\xi}^s = 0, \tag{4}$$

where the linear operator $\mathcal{W}$ is given by

$$\mathcal{W}\boldsymbol{\xi}^s = \boldsymbol{\nabla}\delta^s p - \frac{1}{\rho}\boldsymbol{\nabla} p\delta^s\rho + \rho\boldsymbol{\nabla}\delta^s U. \tag{4a}$$

3. Matrix form of equations of motion

Equation (4) in its integro-differential form is highly complicated. We convert it into a set of linear algebraic equations by expanding $\boldsymbol{\xi}^s$ in terms of a complete set of known basis vectors, $\{\boldsymbol{\zeta}^r(\mathbf{x})\}$. Thus

$$\boldsymbol{\xi}^s = \boldsymbol{\zeta}^r Z^{rs}, \tag{5}$$

where Z^{rs} are the expansion coefficients and, as yet, are unknown. Associated with this, we define the following matrices

$$S^{rs} = \int \rho\,\boldsymbol{\zeta}^{r*}\cdot\boldsymbol{\zeta}^s \mathrm{d}^3x\,, \tag{6a}$$

$$\begin{aligned}C^{rs} &= +i/\Omega\int\rho\,\boldsymbol{\zeta}^{r*}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\zeta}^s)\mathrm{d}^3x \\ &= -i\boldsymbol{\Omega}/\Omega\cdot\int\rho\,\boldsymbol{\zeta}^{r*}\times\boldsymbol{\zeta}^s\mathrm{d}^3x,\end{aligned} \tag{6b}$$

$$W^{rs} = \int \boldsymbol{\zeta}^{r*}\cdot\mathcal{W}\boldsymbol{\zeta}^s\mathrm{d}^3x, \tag{6c}$$

where the integration is over the volume of the star. All these matrices are Hermitian. Furthermore, let $Z = [Z^{rs}]$ be the matrix of the expansion coefficients, and $\omega = [\omega^s\delta^{rs}]$ be the diagonal matrix of the eigenvalues. Substituting Eq. (5) in Eq. (4), multiplying the resulting equation from left by $\boldsymbol{\zeta}^{q*}$, say, and integrating over the volume of the star gives the $(q\,s)$ element of the following matrix equation

$$WZ + 2\Omega CZ\omega - SZ\omega^2 = 0. \tag{7}$$

It is important to note that all factors in Eq. (7) are matrices and should not be commuted with each other without due care. Solutions of Eq. (7) are equivalent to those of Eq. (4). The eigenfrequencies ω in both equations are the same, and once the eigenmatrix Z is known, the set of eigendisplacement vectors $\{\boldsymbol{\xi}^s\}$ can be constructed by Eq. (5).

4. Partitioning into poloidal and toroidal fields

The basis set $\{\zeta^r\}$ can be divided into two poloidal and toroidal subsets, $\{\boldsymbol{\zeta}_{\rm p}^r \mid \boldsymbol{\zeta}_{\rm t}^r\}$. The set of the eigenmodes, $[\boldsymbol{\xi}^s]$, of Eq. (4) in the absence of rotation also has such exact partitioning. Its poloidal subset comprises the commonly-known g- and p-modes of the fluid. The toroidal subset includes those displacements of the fluid which do not perturb the equilibrium state of the star. For them, $\omega = 0$. For the sake of mathematical completeness one might say that $\omega = 0$ is a degenerate eigenfrequency and the set of all toroidal motions $[\boldsymbol{\xi}_{\rm t}^s]$ are its eigendisplacement vectors.

In the presence of rotation, two things happen. a) Each known poloidal mode of the fluid acquires a small toroidal component. b) The neutral toroidal displacements organize themselves into a new sequence of modes and the degeneracy of $\omega = 0$ gets removed. Nonetheless, one may still partition the eigensets $[\boldsymbol{\xi}^s]$ as $[\boldsymbol{\xi}_{\rm p}^s \mid \boldsymbol{\xi}_{\rm t}^s]$ remembering that the subsets $[\boldsymbol{\xi}_{\rm p}^s]$ and $[\boldsymbol{\xi}_{\rm t}^s]$, unlike the no rotation case, are only predominantly poloidal and toroidal, respectively. In view of these considerations, Eq. (5) partitions as

$$\boldsymbol{\xi}_{\rm p}^s = \boldsymbol{\zeta}_{\rm p}^r Z_{\rm pp}^{rs} + \boldsymbol{\zeta}_{\rm t}^r Z_{\rm tp}^{rs}, \tag{8a}$$

$$\boldsymbol{\xi}_{\rm t}^s = \boldsymbol{\zeta}_{\rm p}^r Z_{\rm pt}^{rs} + \boldsymbol{\zeta}_{\rm t}^r Z_{\rm tt}^{rs}, \tag{8b}$$

or in its matrix form and suppressing the superscripts one gets

$$[\boldsymbol{\xi}_{\rm p} \mid \boldsymbol{\xi}_{\rm t}] = [\boldsymbol{\zeta}_{\rm p} \mid \boldsymbol{\zeta}_{\rm t}] \begin{bmatrix} Z_{\rm pp} & Z_{\rm pt} \\ Z_{\rm tp} & Z_{\rm tt} \end{bmatrix}. \tag{8c}$$

Accordingly, all matrices in Eqs. (6) partition into four pp, pt, tp, and tt blocks. For example, an element $S_{\rm pt}^{rs}$ in the pt block is obtained by inserting the two vector $\boldsymbol{\zeta}_{\rm p}^r$ and $\boldsymbol{\zeta}_{\rm t}^s$ in Eq. (6a) and carrying out the integration.

We are not interested in the poloidal modes of Eq. (8a). They are discussed in ample detail and in a much wider scope than that of the present work in Sobouti (1980). Here we concentrate on the toroidal modes of Eq. (8b). The required information comes from multiplying the block partitioned forms of all the matrices in Eq. (7) and extracting the tt and pt blocks of it. Thus

$$\begin{aligned} &\text{tt}-\text{block of Eq. (7)}: \\ &W_{\rm tt}Z_{\rm tt} + 2\Omega C_{\rm tt}Z_{\rm tt}\omega_{\rm t} - S_{\rm tt}Z_{\rm tt}\omega_{\rm t}^2 \\ &+W_{\rm tp}Z_{\rm pt} + 2\Omega C_{\rm tp}Z_{\rm pt}\omega_{\rm t} - S_{\rm tp}Z_{\rm pt}\omega_{\rm t}^2 = 0, \end{aligned} \tag{9a}$$

$$\begin{aligned} &\text{pt}-\text{block of Eq. (7)}: \\ &W_{\rm pt}Z_{\rm tt} + 2\Omega C_{\rm pt}Z_{\rm tt}\omega_{\rm t} - S_{\rm pt}Z_{\rm tt}\omega_{\rm t}^2 \\ &+W_{\rm pp}Z_{\rm pt} + 2\Omega C_{\rm pp}Z_{\rm pt}\omega_{\rm t} - S_{\rm pp}Z_{\rm pt}\omega_{\rm t}^2 = 0. \end{aligned} \tag{9b}$$

5. Partitioning by spherical harmonic numbers

For a toroidal basis vector we will adopt the following spherical harmonic form

$$\begin{aligned} \boldsymbol{\zeta}_{\rm t}^{r\ell} &= \boldsymbol{\nabla} \times [\hat{r}\phi^{r\ell}(r)Y_\ell^m(\vartheta,\varphi)] \\ &= \frac{\phi^{r\ell}}{r}\left(0, \frac{im}{\sin\vartheta}Y_\ell^m, -\frac{\partial Y_\ell^m}{\partial\vartheta}\right). \end{aligned} \tag{10a}$$

For a poloidal vector we will take

$$\boldsymbol{\zeta}_{\rm p}^{r\ell} = \frac{1}{r}\left[\psi^{r\ell}(r)Y_\ell^m, \chi^{r\ell}(r)\frac{\partial Y_\ell^m}{\partial\vartheta}, \chi^{r\ell}(r)\frac{im}{\sin\vartheta}Y_\ell^m\right]. \tag{10b}$$

An appropriate ansatz for the radial function ϕ, ψ, and χ are given in Appendix B. The toroidal vector (10a) is obviously derived from a radial vector potential. The poloidal vector (10b) is actually the sum of two vectors, one derived from a scalar potential and the other derived from a toroidal vector potential. See Sobouti (1981).

In Eq. (10) we have suppressed the harmonic index m from $\boldsymbol{\zeta}$'s because a slowly rotating star is axially symmetric. Vectors with different values of m are not mutually coupled. Vectors belonging to the same m, but different ℓ's, however, are coupled. This feature entails a further partitioning of the basis sets into their harmonic subsets, $[\boldsymbol{\zeta}_{\rm p}^\ell,\ \ell = 0, 1, 2, \cdots]$ and $[\boldsymbol{\zeta}_{\rm t}^\ell,\ \ell = 1, 2, \cdots]$. Correspondingly, each of the matrices in Eqs. (9) partitions into blocks, designated by a pair of harmonic numbers (k, ℓ), say. For example, the rs element of $S_{\rm pt}^{k\ell}$, say, will be obtained from Eq. (6a) by inserting the two vectors $\boldsymbol{\zeta}_{\rm p}^{rk^*}$ and $\boldsymbol{\zeta}_{\rm t}^{s\ell}$ in that equation. In the following we rewrite Eqs. (9) taking into account the new partitioning. Thus,

tt-block:

$$\begin{aligned} \sum_{\ell'} \Big[&W_{\rm tt}^{k\ell'}Z_{\rm tt}^{\ell'\ell} + 2\Omega C_{\rm tt}^{k\ell'}Z_{\rm tt}^{\ell'\ell}\omega_{\rm t}^\ell - S_{\rm tt}^{k\ell'}Z_{\rm tt}^{\ell'\ell}\omega_{\rm t}^{\ell 2} \\ &+W_{\rm tp}^{k\ell'}Z_{\rm pt}^{\ell'\ell} + 2\Omega C_{\rm tp}^{k\ell'}Z_{\rm pt}^{\ell'\ell}\omega_{\rm t}^\ell - S_{\rm tp}^{k\ell'}Z_{\rm pt}^{\ell'\ell}\omega_{\rm t}^{\ell 2} \Big] = 0, \end{aligned} \tag{11a}$$

pt-block:

$$\begin{aligned} \sum_{\ell'} \Big[&W_{\rm pt}^{k\ell'}Z_{\rm tt}^{\ell'\ell} + 2\Omega C_{\rm pt}^{k\ell'}Z_{\rm tt}^{\ell'\ell}\omega_{\rm t}^\ell - S_{\rm pt}^{k\ell'}Z_{\rm tt}^{\ell'\ell}\omega_{\rm t}^{\ell 2} \\ &+W_{\rm pp}^{k\ell'}Z_{\rm pt}^{\ell'\ell} + 2\Omega C_{\rm pp}^{k\ell'}Z_{\rm pt}^{\ell'\ell}\omega_{\rm t}^\ell - S_{\rm pp}^{k\ell'}Z_{\rm pt}^{\ell'\ell}\omega_{\rm t}^{\ell 2} \Big] = 0. \end{aligned} \tag{11b}$$

We again emphasize that each of the factors in Eqs. (11) are matrices in their own right.

6. Expansion order and spherical harmonic structure of the various matrices

Expansions of ρ, p, and U in powers of Ω^2 results in a corresponding expansion of all the matrices in Eqs. (11). Moreover, having the spherical harmonics forms of Eqs. (10), integrations over ϑ and φ dependencies in the calculation of matrix elements can be performed analytically. These two tasks are carried out in Appendix A. The results are quoted below:

$$S_{\rm tt}^{k\ell} = S_{0\rm tt}^{\ell\ell}\delta_{k\ell} + \Omega^2 S_{2\rm tt}^{k\ell}(\delta_{k\ell} + \delta_{k,\ell\pm2}), \quad \text{see (A.7)}-\text{(A.8)}, \tag{12}$$

$$C_{\rm tt}^{k\ell} = C_{0\rm tt}^{\ell\ell}\delta_{k\ell} + \Omega^2 C_{2\rm tt}^{k\ell}(\delta_{k\ell} + \delta_{k,\ell\pm2}), \quad \text{see (A.9)}-\text{(A.10)} \tag{13a}$$

$$C_{0\rm tt}^{\ell\ell} = (m/\ell(\ell+1))S_{0\rm tt}^{\ell\ell}, \quad \text{see (A.9)}, \tag{13b}$$

$$C_{\rm tp}^{k\ell} = C_{\rm pt}^{\ell k^*} = C_{0\rm tp}^{k\ell}\delta_{k\ell\pm1} + \mathcal{O}(\Omega^2), \quad \text{see (A.11)}, \tag{13c}$$

$$W_{\rm pp}^{k\ell} = W_{0\rm pp}^{\ell\ell}\delta_{k\ell} + \mathcal{O}(\Omega^2), \quad \text{see (A.12)}, \tag{14a}$$
$$W_{\rm tp}^{k\ell} = W_{\rm pt}^{\ell k *} = \Omega^2 W_{2\rm tp}^{k\ell}\delta_{k\ell\pm1}, \quad \text{see (A.13)}, \tag{14b}$$
$$W_{\rm tt}^{k\ell} = \Omega^4 W_{4\rm tt}^{k\ell}\left(\delta_{k\ell} + \delta_{k,\ell\pm2}\right), \quad \text{see (A.14)}. \tag{14c}$$

A subscript, 0, 2, 4 preceding the tt, tp, pt or pp designations of the matrices indicates the order of Ω in the matrix in question. We also note that as $\Omega \to 0$, $\omega_{\rm t}$ does the same. Therefore, it must have the following form

$$\omega_{\rm t}^{\ell} = \Omega\left(\omega_{0\rm t}^{\ell} + \Omega^2\omega_{2\rm t}^{\ell}\right). \tag{15}$$

Substituting these order of magnitude informations in Eq. (11b) reveals that

$$Z_{\rm pt}^{k\ell} = \Omega^2 Z_{2\rm pt}^{k\ell}. \tag{16a}$$

For $Z_{\rm tt}^{k\ell}$, we have no information so far. Therefore, we assume the general form

$$Z_{\rm tt}^{k\ell} = Z_{0\rm tt}^{k\ell} + \Omega^2 Z_{2\rm tt}^{k\ell}. \tag{16b}$$

7. Expansion of equations of motion

Equations (12)–(16) allow a consistent expansion of Eqs. (11) at Ω^2 and Ω^4 orders and enable one to decipher the information contained in them.

7.1. Ω^2 order of the tt-block

At Ω^2 order Eq. (11a) gives

$$2C_{0\rm tt}^{kk}Z_{0\rm tt}^{k\ell} - S_{0\rm tt}^{kk}Z_{0\rm tt}^{k\ell}\omega_{0t}^{\ell} = 0. \tag{17}$$

For $k = \ell$, considering the proportionality of $C_{0\rm tt}^{\ell\ell}$ and $S_{0\rm tt}^{\ell\ell}$, Eq. (13b), one obtains

$$\omega_{0\rm t}^{\ell} = 2m/\ell(\ell+1)I, \quad I = \text{unit matrix}, \tag{18a}$$
$$Z_{0\rm tt}^{\ell\ell} \quad \text{undetermined at this stage.} \tag{18b}$$

For $k \neq \ell$, using Eqs. (13b) and (18a), one has

$$2m[1/k(k+1) - 1/\ell(\ell+1)]S_{0\rm tt}^{kk}Z_{0\rm tt}^{k\ell} = 0 \text{ or } Z_{0\rm tt}^{k\ell} = 0. \tag{18c}$$

Let us summarize the findings so far from a pedagogical point of view. At the Ω^2 order one solves the eigenvalue Eq. (17). In this equation the Coriolis forces remove the degeneracy of the neutral motions and create a sequence of modes of purely toroidal nature. The new modes have a definite ℓ-symmetry, (Eqs. (18)). They are not coupled with toroidal motions of $k \neq \ell$ symmetry (that is $Z_{0\rm tt}^{k\ell} = 0$) and with poloidal motion (that is $Z_{0\rm pt}^{k\ell} = 0$, see Eq. (16a)). Removal of degeneracy, however, is partial. For, of the three designations of a standing wave in three dimensions, only (ℓ, m) designations have appeared in the expression for $\omega_{0\rm t}^{\ell} = 2m/\ell(\ell+1)$. A third designation, indicating the radial wave number, is as yet absent. They will appear at higher orders of Ω.

One simplifying feature: We note that $\omega_{0\rm t}^{\ell}$ of Eq. (18a) is a constant matrix. Therefore, it will commute with any matrix carrying the same $\ell\ell$ designations, such as $Z_{0\rm tt}^{\ell\ell}$, $S_{0\rm tt}^{\ell\ell}$, etc. This feature was used in the derivation of Eq. (18c) and will be used repeatedly in what follows to simplify the matrix manipulations.

7.2. Ω^2 order of pt-block

Equation (11b) at order Ω^2 along with Eqs. (18) gives

$$W_{0\rm pp}^{kk}Z_{2\rm pt}^{k\ell} + (W_{2\rm pt}^{k\ell} + 2C_{0\rm pt}^{k\ell}\omega_{0\rm t}^{\ell})Z_{0\rm tt}^{\ell\ell} = 0, k = \ell \pm 1. \tag{19a}$$

$W_{0\rm pp}^{kk}$ is associated with poloidal modes of the non-rotating fluids. In fact if $\boldsymbol{\zeta}_{\rm p}$ are chosen to be the exact eigenvectors of the non-rotating star, $W_{\rm pp}^{kk}$ will be a diagonal matrix whose elements are the square of the eigenfrequencies of the p- and g-modes. At least in the case of p-modes, $W_{0\rm pp}^{kk}$ is invertible (see Sobouti 1980 for complications in the case of g-modes). Thus, one obtains

$$Z_{2\rm pt}^{k\ell} = -\left(W_{0\rm pp}^{kk}\right)^{-1}\left[W_{2\rm pt}^{k\ell} + C_{0\rm pt}^{k\ell}\omega_{0\rm t}^{\ell}\right]Z_{0\rm tt}^{\ell\ell}, k = \ell \pm 1. \tag{19b}$$

By Eq. (8b), Eq. (19b) expresses that a toroidal mode of the rotating fluid of ℓ symmetry acquires a small poloidal component of $\ell \pm 1$ symmetry at Ω^2 order.

The case of g-modes is different. Rotation, no matter how small, cannot be treated as a perturbation on them, as they are created by minute buoyancy forces and the low frequency tail of their spectrum gets completely wiped out by any other force in the medium, here the Coriolis forces. This, in mathematical language, means that $W_{0\rm pp}$ for g-modes is not invertible and Eq. (19b) is not applicable to them. The way out of the dilemma is to consider the sum of buoyancy and other intruding forces as an inseparable entity, without dividing it to large and small components. The works of Provost et al. (1980) and of Sobouti (1977, 1980) are examples of such treatments. We leave it to the experts in the field to decide whether the scrutiny of g-modes in rotating neutron stars is a crucial or an irrelevant issue.

7.3. Ω^4 order of tt-block

7.3.1. The $\ell\ell$-subblock

A systematic extraction of Ω^4 order terms of the $\ell\ell$-block of Eq. (11a) with the help of Eqs. (12)–(16) and elimination of $Z_{2\rm pt}^{k\ell}$ term appearing in it by Eq. (19b) gives

$$T_{4\rm tt}^{\ell\ell}Z_{0\rm tt}^{\ell\ell} - \frac{2m}{\ell(\ell+1)}S_{0\rm tt}^{\ell\ell}Z_{0\rm tt}^{\ell\ell}\omega_{2\rm t}^{\ell} = 0, \tag{20}$$

where the fourth order T-matrix is

$$T_{4\rm tt}^{\ell\ell} = [W_{4\rm tt} + (2C_{2\rm tt} - S_{2\rm tt}\omega_{0\rm t})\omega_{0\rm t} - (W_{2\rm tp} + 2C_{0\rm tp}\omega_{0\rm t})W_{0\rm pp}^{-1}(W_{2\rm pt} + 2C_{0\rm pt}\omega_{0\rm t})]^{\ell\ell}. \tag{20a}$$

Equation (20) is a simple eigenvalue problem. Vanishing of its characteristic determinant,

$$\left|T_{4\rm tt}^{\ell\ell} - \frac{2m}{\ell(\ell+1)}S_{0\rm tt}^{\ell\ell}\omega_{2\rm t}^{\ell}\right| = 0, \tag{20b}$$

will give the non-degenerate second order eigenvalues $\omega_{2\rm t}^{s\ell}$, s = radial wave number. Once they are known, Eq. (20) itself can be solved for the eigenmatrix $Z_{0\rm tt}^{\ell\ell}$. We note that we have solved two eigenvalue problems, Eqs. (17) and (20), to remove all degeneracies of the zero frequency toroidal motions of the non-rotating fluid. Further extension of the analysis of equations of motion to orders higher than Ω^4 will result in non-homogeneous algebraic equations whose non-homogeneous terms are given in terms of the matrices calculated in the previous orders.

7.3.2. The $(\ell, \ell \pm 2)$-subblock

The presence of $\delta_{k,\ell\pm2}$ in Eqs. (13a) and (14c) indicates that two toroidal motions belonging to ℓ and $\ell \pm 2$ are mutually coupled. Likewise, the presence of $\delta_{k,\ell\pm1}$ in Eqs. (13c) and (14b) shows the coupling of toroidal and poloidal motion of ℓ and $\ell \pm 1$ symmetry. This brings in an additional coupling between two toroidal motions of ℓ and $\ell \pm 2$ symmetries through the intermediary of poloidal motions. Therefore, the only unexplored blocks of Eq. (11a) are those with $(\ell, \ell \pm 2)$ designations. As in Sect. 7.3.1 above, we extract the Ω^4 order terms of Eq. (11a), but this time with ℓ, $\ell \pm 2$ superscripts, eliminate $Z_{2\text{pt}}^{\ell',\ell'\pm1}$ appearing in it by Eq. (19b) and arrive at

$$T_{4\text{tt}}^{\ell,\ell\pm2} Z_{0\text{tt}}^{\ell\pm2,\ell\pm2} = \Big[2C_{0\text{tt}}^{\ell\ell} Z_{2\text{tt}}^{\ell,\ell\pm2} - S_{0\text{tt}}^{\ell\ell} Z_{2\text{tt}}^{\ell,\ell\pm2} \omega_{0\text{t}}^{\ell\pm2}\Big] \omega_{0\text{t}}^{\ell\pm2}, \quad (21)$$

where

$$T_{4\text{tt}}^{\ell,\ell\pm2} = \Bigg[W_{4\text{tt}} + (2C_{2\text{tt}} - S_{2\text{tt}}\omega_{0\text{t}})\,\omega_{0\text{t}} - \left(W_{2\text{tp}} + \frac{4m}{(\ell\pm2)(\ell\pm2+1)} C_{0\text{tp}}\right) W_{0\text{pp}}^{-1} \times \left(W_{2\text{pt}} + \frac{4m}{(\ell\pm2)(\ell\pm2+1)} C_{0\text{pt}}\right)\Bigg]^{\ell,\ell\pm2}. \quad (21a)$$

In deriving this expression, on two occasions we have substituted for $\omega_{0\text{t}}^{\ell\pm2}$ and shifted the scalar factor $2m/(\ell \pm 2)(\ell \pm 2 + 1)$ across the other matrices. Returning to Eq. (21) we substitute for $C_{0\text{tt}}^{\ell\ell}$ from Eq. (13b) and solve for $Z_{2\text{tt}}$. Thus,

$$Z_{2\text{tt}}^{\ell,\ell\pm2} = \left[\left(\omega_{0\text{t}}^{\ell} - \omega_{0\text{t}}^{\ell\pm2}\right)\omega_{0\text{t}}^{\ell\pm2}\right]^{-1} \times (S_{0\text{tt}}^{\ell\ell})^{-1} T^{\ell,\ell\pm2} Z_{0\text{tt}}^{\ell\pm2,\ell\pm2}. \quad (22)$$

Equations (20), (21) and (22) are all the information contained in the Ω^4 order of the tt-block. Whether $Z_{2\text{tt}}^{\ell,\ell}$ is non-zero or otherwise is not clear at this level. To answer the question one has to go to Ω^4 and Ω^6 orders of Eqs. (11b) and (11a), respectively. This, however, will not be attempted here.

7.4. Bibliographical notes

Papaloizou & Pringle (1978) have studied the low frequency g- and r-modes of Eq. (4) in an equipotential coordinate system with their applicability to the short period oscillations of cataclysmic variables in mind.

Sobouti (1980) has studied the problem primarily with the goal of analyzing the perturbative effects of slow rotations on p-modes and demonstrating that rotation, no matter how small, cannot be treated as a perturbation on g-modes. He argues that the g-modes are fragile structures and their low frequency tail of the spectrum, below the rotation frequency of the star, will be completely wiped out by Coriolis and asphericity forces of the star. The criterion for the validity of perturbation expansion is that the perturbing operator should be smaller than the initial unperturbed operator everywhere in the Hilbert space spanned by the eigenfunctions of the unperturbed operator, (Rellich 1969). This condition is not met by g-modes when exposed to rotation, magnetic, tidal forces, etc., as they have vanishingly small eigenfrequencies.

Provost et al. (1981) present an analysis of what they call the "quasi-toroidal modes of slowly rotating stars". Their work should be noted for the consistency of mathematical manipulations exercised throughout the paper. They noted that in neutrally convective rotating stars one cannot have modes with predominantly toroidal motions. They get mixed with the neutral convective displacements.

Lockitch & Friedman (1999) also address the hybrid modes with comparable toroidal and poloidal motions. Their work should be noted for the emphasis put on the (ℓ, m) parities of the hybrid components that get coupled through asphericity forces.

Yoshida & Lee (2000) study Eq. (4) for a) those modes that are predominantly toroidal in their nature and b) for those that have comparable poloidal and toroidal components. Their latter modes are the same as those of Provost et al. (1981).

8. Numerical result for rotating polytropes

The bulk properties of some observed neutron stars seem to approximate those of a polytrope of index 1. See Sterigioulas (1998). It has become fashionable to categorize neutron stars as stiff or soft, depending on whether their density distributions are similar to those of polytropes of index smaller than 1.5 or larger, respectively. To have an example of each category, sample calculations are given for polytropes of indices 1 and 2.

The structure of rotating polytropes, taken from Chandrasekhar (1933), is summarized in Appendix C. The ansatz for the scalars $\phi^{s\ell}$, $\psi^{s\ell}$ and $\chi^{s\ell}$ appearing in Eqs. (10) are given in Appendix B. The required matrix elements are reduced in Appendix A. For a given $N = 1, 2, \ldots$, the $N \times N$ matrices are numerically integrated. The eigenvalues $\omega_{0\text{t}}^{\ell}$ and $\omega_{2\text{t}}^{\ell}$ are calculated from Eqs. (18a) and (20b). Once the eigenvalues are known, the various components of the eigenmatrices $Z_{0\text{tt}}^{\ell\ell}$, $Z_{2\text{tt}}^{\ell\pm2,\ell}$ and $Z_{2\text{pt}}^{\ell\pm1,\ell}$ are calculated from Eqs. (20), (22) and (19b). The results are given in Tables 1 to 3.

In Table 1, to show the convergence of the variational calculations, the eigenvalues $\omega_{2\text{t}}^{\ell}$ are displayed for a polytrope of index 2, $\ell = m = 2$, and for $N = 1, 2, 3, 4, 5$. This table should be considered as a basis for judging the accuracy of the numerical values. With only five variational parameters, the first, second and third eigenvalues are produced with an accuracy of a few parts in 10^5, 10^4

Table 1. Convergence of the variational calculations. For polytrope of index 2 and $\ell = m = 2$, the second order toroidal eigenvalues, $\omega^{\ell}_{2\mathrm{t}}$, are displayed for different number of variational parameters, $N = 1, 2, 3, 4, 5$. They are in units of $\sqrt{\pi G\bar{\rho}}$. A number $a \times 10^{\pm b}$ is written as $a \pm b$.

.30169+0				
.19860+0	.60797+0			
.16137+0	.44344+0	.1371+1		
.16136+0	.43726+0	.11361+1	.30568+1	
.16132+0	.43714+0	.10106+1	.24613+1	.52712+1

and 10^2, respectively. Likewise, the numerical values in the remaining tables should be trusted to the same degree of accuracy and for the first few modes.

In Table 2, the second order eigenvalues, $\omega^{\ell}_{2\mathrm{t}}$ and coefficient matrices, $Z^{\ell\ell}_{0\mathrm{tt}}$, $Z^{\ell+1,\ell}_{2\mathrm{pt}}$, and $Z^{\ell+2,\ell}_{2\mathrm{tt}}$ are given for polytrope of index 1 and for $\ell = m = 2, N = 5$. In Table 3 we give the same calculations for polytrope of index 2. The eigenvalues are in units of $\sqrt{\pi G\bar{\rho}}$. They are in agreement with those of Lindblom et al. (1999), Yoshida & Lee (2000), and Morsink (2001). Each of these authors have used their own technique which are different from that of the present paper.

A novel feature of the present analysis is the provision of much detail on eigendisplacement vectors, information that can be profitably used to calculate any other bulk or local parameter of the model. For example, for modes belonging to $\ell = m = 2$ one may write

$$\boldsymbol{\xi}^{s,2}_{\mathrm{t}} = \sum_r \boldsymbol{\zeta}^{r,2}_{\mathrm{t}} (Z^{2,2}_{0\mathrm{tt}})^{rs} + \Omega^2 \sum_r \boldsymbol{\zeta}^{r,3}_{\mathrm{p}} (Z^{3,2}_{2\mathrm{pt}})^{rs} + \Omega^2 \sum_r \boldsymbol{\zeta}^{r,4}_{\mathrm{t}} (Z^{4,2}_{2\mathrm{tt}})^{rs}, \qquad (23)$$

in which the first sum is the backbone of the mode and is of zero order. It is toroidal motion of $\ell = m = 2$ symmetry. The second sum is the coupling of $\ell = 3$, $m = 2$ poloidal motion with $\ell = m = 2$ toroidal motion. It is a poloidal motion and is of second order. The third sum is the coupling of $\ell = 4$, $m = 2$ toroidal motion with $\ell = m = 2$ toroidal motion and again is of second order. In Figs. 1 to 4 we have plotted the radial behavior of $\sum_r \boldsymbol{\zeta}^{r,2}_{\mathrm{t}} (Z^{2,2}_{0\mathrm{tt}})^{rs}$, $\sum_r \boldsymbol{\zeta}^{r,3}_{\mathrm{p}} (Z^{3,2}_{2\mathrm{pt}})^{rs}$, and $\sum_r \boldsymbol{\zeta}^{r,4}_{\mathrm{t}} (Z^{4,2}_{2\mathrm{tt}})^{rs}$ for polytropes of indices 1, 2 and the first two modes, $s = 1, 2$. The center and the surface are nodes in all curves. For $s = 2$ there is an extra node in between in every curve. The general rule is the number of nodes for any parameter $f(r, \theta, \varphi)$ belonging to the radial mode number s, is $s + 1$. This includes the ever-present nodes at the center and the surface of the star. This feature is faithfully present in all five modes that can be constructed from the data of Tables 1 and 2, even though we know that the numerical values for $s = 4$ and 5 are only orders of magnitude.

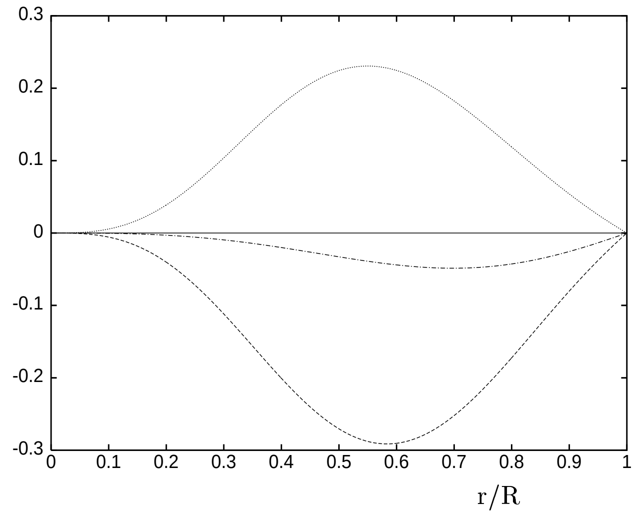

Fig. 1. Radial behavior of the various components of the eigenfunction of Eq. (23) for $\ell = m = 2, s = 1, n = 1$. $\sum_r \boldsymbol{\zeta}^{r,2}_{\mathrm{t}} (Z^{2,2}_{0\mathrm{tt}})^{rs}$, dashed curve; $\sum_r \boldsymbol{\zeta}^{r,3}_{\mathrm{p}} (Z^{3,2}_{0\mathrm{pt}})^{rs}$, dot-dashed curve; $\sum_r \boldsymbol{\zeta}^{r,4}_{\mathrm{t}} (Z^{4,2}_{2\mathrm{pt}})^{rs}$, dotted curve. Data for Z's are taken from the first column of Table 2. Nodes in all three components are at the center and surface.

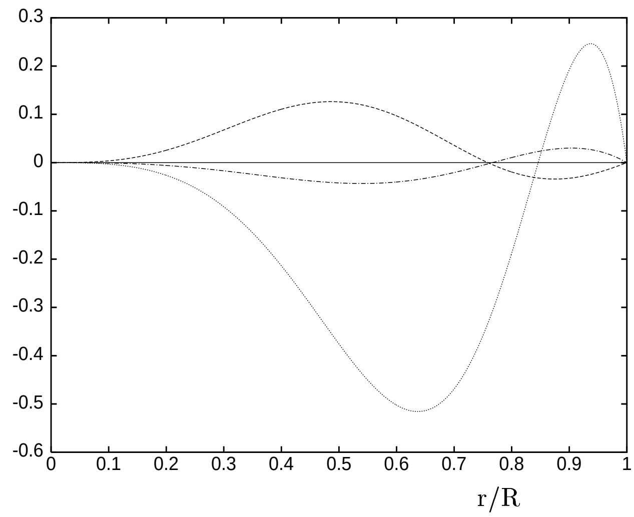

Fig. 2. Same as Fig. 1 for $\ell = m = 2, s = 2, n = 1$. Data for Z's are from the second column of Table 2. Note the extra node in all three components.

9. *r*-mode time scales

In this section we study the dissipative effects of viscosity and gravitational radiation on r-modes. Quite generally and regardless of whether the star rotates or not, the total energy of an undamped mode, $\boldsymbol{\xi}(r, t)$, is

$$E = \frac{1}{2} \int \left[\rho \dot{\boldsymbol{\xi}}_{\mathrm{real}} \cdot \dot{\boldsymbol{\xi}}_{\mathrm{real}} + \boldsymbol{\xi}_{\mathrm{real}} \cdot \mathcal{W} \boldsymbol{\xi}_{\mathrm{real}} \right] \mathrm{d}^3 x, \qquad (24)$$
$$= \textit{maximum kinetic energy of the mode.}$$

See Appendix D for proof of Eq. (24). In the presence of viscous forces and the gravitational radiation, the

Table 2. Second order eigenvalues $\omega_{2\mathrm{t}}$ and coefficient matrices, $Z^{2,2}_{0\mathrm{tt}}$, $Z^{3,2}_{2\mathrm{pt}}$, and $Z^{4,2}_{2\mathrm{tt}}$ are given for $m = 2, N = 5$ and the polytropic index 1.

$\omega_{2\mathrm{t}}$	.32203+0	.84087+0	.19790+1	.48110+1	.96363+1
$Z^{2,2}_{0\mathrm{tt}}$					
	−.58565+1	.38532+1	−.29986+1	−.27754+1	.14743+2
	.12510+2	−.11134+2	.82347+1	.72343+1	−.73216+2
	−.12234+2	.84713+1	−.34999+1	.18385+0	.14699+3
	.64179+1	−.87049+0	−.73634+2	−.15379+2	−.13539+3
	−.15465+1	−.78018+0	.49263+1	.11366+2	.47334+2
$Z^{3,2}_{2\mathrm{pt}}$					
	−.39677+0	−.82952+0	.96949-1	−.12394+0	−.27257+0
	−.20041+0	.98613+0	−.28392+0	.65146+0	.10959+1
	.63624+0	.96043+0	−.27949+0	−.14499+1	−.17471+1
	−.41126+0	−.37431+0	.42798+0	.10202+1	.98804+0
	.92562-1	−.53391-1	−.13166+0	−.23806+0	−.18318+0
$Z^{4,2}_{2\mathrm{tt}}$					
	.58000+1	−.30934+1	.27352+1	.30368+1	−.40984+1
	−.16462+2	−.10961+2	−.10022+2	−.11591+2	.18955+2
	.21461+2	.15392+2	.14924+2	.18085+2	−.35467+2
	−.14090+2	.10291+2	−.10129+2	−.12810+2	.30614+2
	.37608+1	−.27390+1	.26484+1	.33165+1	−.10091+2
	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$

Table 3. Same as Table 2, for polytrope 2.

$\omega_{2\mathrm{t}}$	.16132+0	.43714+0	.10106+1	.24613+1	.52712+1
$Z^{2,2}_{0\mathrm{tt}}$					
	.30158+1	.17867+1	.11589+1	.82841+0	.10498+2
	−.36757+1	−.25829+1	−.20167+0	.15548+1	−.45626+2
	.16613+1	−.30333+1	−.85360+1	−.14096+2	.84738+2
	−.37105+0	.52330+1	.14253+2	.24110+2	−.75082+2
	.14113+0	−.18708+1	−.59865+1	−.13019+2	.25924+2
$Z^{3,2}_{2\mathrm{pt}}$					
	−.14392+1	−.28790-1	.29383+0	.19977-1	−.24544+0
	.10270+1	−.14184+1	−.11345+1	.15290+0	.92290+0
	−.28306+0	.15325+1	.57987+0	-.95505+0	−.14964+1
	−.17515-1	.71869+0	−.63417-1	.68145+0	.73244+0
	.20617-1	.13678+0	−.15707-1	−.15094+0	−.11366+0
$Z^{4,2}_{2\mathrm{tt}}$					
	−.30726+1	− .16563+1	−.14505+1	−.14874+1	−.21505+1
	.71922+1	.48149+1	.43706+1	.46605+1	.81784+1
	−.83777+1	−.58904+1	−.56669+1	−.63205+1	−.13491+2
	.51958+1	.36393+1	.34767+1	.40067+1	.10782+2
	−.13588+1	.93612+0	−.85413+0	−.92167+0	−.34032+1
	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$

combined rate of dissipation (Cutler & Lindblom 1987) is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int(2\eta\delta\sigma^*_{ab}\delta\sigma^{ab} + \zeta\delta\sigma^*\delta\sigma)\mathrm{d}^3x - \omega_\mathrm{t}(\omega_\mathrm{t} - m\Omega)\sum_{\ell\geq 2} N_\ell(\omega_t - m\Omega)^{2\ell} \times \left(|\delta D_{\ell m}|^2 + |\delta J_{\ell m}|^2\right), \quad (25)$$

where η and ζ are the shear and bulk viscosities respectively,

$$\delta\sigma_{ab} = \tfrac{1}{2}(\nabla_a\dot{\xi}_b + \nabla_b\dot{\xi}_a - \tfrac{2}{3}\delta_{ab}\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\xi}}), \quad \text{shear strain}, \quad (26a)$$

$$\delta\sigma = \nabla\cdot\dot{\boldsymbol{\xi}}, \quad \text{bulk strain}, \quad (26b)$$

$$N_\ell = \frac{4\pi G}{c^{2\ell+1}}\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)[(2\ell+1)!!]^2}, \quad c = \text{ speed of light},$$

$$\delta D_{\ell m} = \int \delta\rho\, r^\ell Y^*_{\ell m}\mathrm{d}^3x, \quad \text{mass multipole moment}, \quad (27)$$

$$\delta J_{\ell m} = \frac{2}{c}\sqrt{\ell/(\ell+1)}\int(\rho\dot{\boldsymbol{\xi}} + \delta\rho\,\mathbf{v})\cdot\boldsymbol{Y}^*_{\ell m}\mathrm{d}^3x, \quad \text{current multipole moment}, \quad (28)$$

$$\boldsymbol{Y}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}}\nabla\times(\hat{\boldsymbol{r}}Y_{\ell m}),$$

toroidal vector spherical harmonics. (28a)

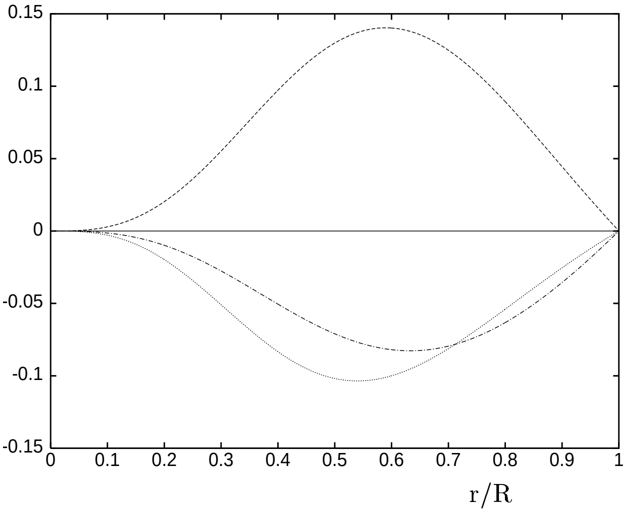

Fig. 3. Same as Fig. 1 for $\ell = m = 2, s = 1, n = 2$. Data for Z's are from the first column of Table 3. Nodes are at the center and the surface.

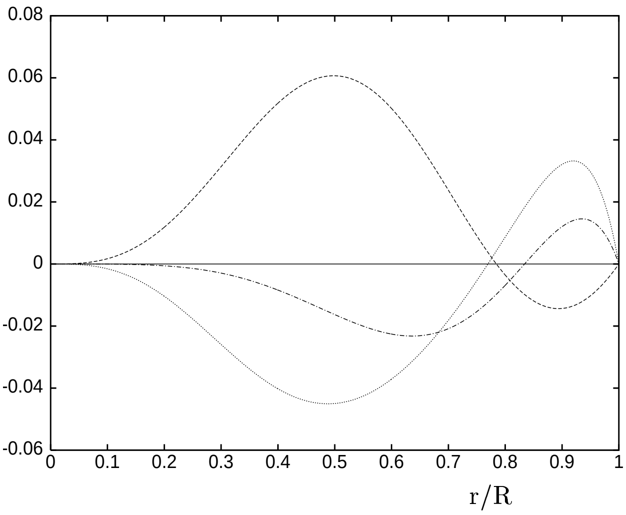

Fig. 4. Same as Fig. 1 for $\ell = m = 2, s = 2, n = 2$. Data for Z's are from the second column of Table 3. Note the extra nodes in all components.

For a mode of the type of Eq. (23) the bulk viscous dissipation (bv) comes entirely from its poloidal component, as $\boldsymbol{\nabla}\cdot\boldsymbol{\zeta}_{\rm t} = 0$. For $m = \ell$ this has the following rate

$$(\mathrm{d}E^{s\ell}/\mathrm{d}t)_{bv} = -\omega_{0\rm t}^{\ell 2}\Omega^6 \left\{ Z_{2\rm pt}^{k\ell\dagger}(bv) Z_{2\rm pt}^{k\ell} \right\}^{ss}, k = \ell + 1 \quad (29)$$

where the elements of the new matrix, (bv), are

$$(bv)^{rq} = \int \zeta \left[\mathrm{d}(r\psi^{rk})/\mathrm{d}r - k(k+1)\chi^{rk}\right] \left[\mathrm{d}(r\psi^{qk})/\mathrm{d}r - k(k+1)\chi^{qk}\right] \mathrm{d}r/r^3. \quad (29\mathrm{a})$$

The shear viscous dissipation has contributions from both toroidal and poloidal components of Eq. (23). To the lowest order in Ω, however, the toroidal component has the dominant contribution. Thus,

$$\left(\mathrm{d}E^{s\ell}/\mathrm{d}t\right)_{\rm sv} = -\omega_{0\rm t}^{\ell 2}\Omega^2 \left\{ Z_{0\rm tt}^{\ell\ell\dagger}(sv) Z_{0\rm tt}^{\ell\ell} \right\}^{ss}, \quad (30)$$

Table 4. Shear viscous-, bulk viscous-, and gravitational radiation- time scales in seconds are given for polytrope 1 and $s = 1, 2, 3, 4, 5$ from top to bottom. A number $a \times 10^{\pm b}$ is written as $a \pm b$.

s	$\tilde{\tau}_{\rm sv}$	$\tilde{\tau}_{\rm bv}$	$\tilde{\tau}_{\rm gr}$
1	$2.14+8$	$1.20+11$	$-2.79+0$
2	$7.78+8$	$1.12+11$	$-8.71-1$
3	$1.41+9$	$1.04+11$	$-5.53-1$
4	$1.72+9$	$9.72+10$	$-5.66-1$
5	$1.47+8$	$9.06+10$	$-1.00+0$

where

$$(sv)^{rq} = \ell(\ell+1)\int \eta \left[(r\mathrm{d}\phi^{r\ell}/\mathrm{d}r - 2\phi^{r\ell}) \times(r\mathrm{d}\phi^{q\ell}/\mathrm{d}r - 2\phi^{q\ell}) +(\ell^2+\ell-2)\phi^{r\ell}\phi^{q\ell}\mathrm{d}r/r^2\right]. \quad (30\mathrm{a})$$

Dissipation due to the gravitational radiation, to the lowest order, comes from the current multipole moment, $\delta^r J_\ell$. This has the following vector elements

$$\delta^r J_\ell = i\omega_{0\rm t}^\ell \Omega \frac{2\ell}{c}\int \rho_0\, r^{\ell+1}\phi^{r\ell}\mathrm{d}r. \quad (31)$$

To obtain the dissipation time scales: a)we have calculated the integrals of Eqs. (24)–(31), numerically. b) For the bulk and shear viscosities we have adopted the values of Cutler and Lindblom, $\eta = 347\rho^{9/4}T^{-2}$ g cm^{-1} s^{-1}, $\zeta = 6.0\times10^{-59}\rho^2\omega_{\rm t}^{-2}T^6$ g cm^{-1} s^{-1}, where T is the temperature. For the a mode of radial node number s and $\ell = m = 2$, we obtain

$$\frac{1}{\tau^s(\Omega,T)} = \frac{1}{\tilde{\tau}^s_{\rm sv}}\left(\frac{10^9\ \mathrm{K}}{T}\right)^2 + \frac{1}{\tilde{\tau}^s_{\rm bv}}\left(\frac{T}{10^9\ \mathrm{K}}\right)^6\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right) + \frac{1}{\tilde{\tau}^s_{\rm gr}}\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)^3, \quad (32)$$

where $\bar{\rho}$ is the average density of the star. Here $\tilde{\tau}^s_{\rm sv}$, $\tilde{\tau}^s_{\rm bv}$, and $\tilde{\tau}^s_{\rm gr}$ are the shear viscous-, bulk viscous- and the gravitational radiation- time scale, respectively, which are normalized for $T = 10^9$ K and $\Omega^2 = \pi G\bar{\rho}$. The total dissipation time scale is

$$\frac{1}{\tau(\Omega,T)} = \sum_s \frac{1}{\tau^s(\Omega,T)}. \quad (33)$$

For a model of 1.4 $M_\odot$ and $R = 12.53$ km we have calculated $\tilde{\tau}^s_{\rm sv}$, $\tilde{\tau}^s_{\rm bv}$, and $\tilde{\tau}^s_{\rm gr}$ for $s = 1, 2, 3, 4, 5$ and displayed in the unit of time in Tables 4 and 5. Our values for $s = 1$ are in agreement with those of Lindblom et al. (1999) and, Yoshida & Lee (2000). To the best of our knowledge, the values for $s \geq 2$ are new.

In a newly born hot neutron star, $T \geq 10^{12}$ K, the bulk viscosity has a dominant role in damping out the perturbations and cooling down the star. In colder stars,

Table 5. Same as Table 4, for polytrope 2.

s	$\tilde{\tau}_{\rm sv}$	$\tilde{\tau}_{\rm bv}$	$\tilde{\tau}_{\rm gr}$
1	$1.28+9$	$2.07+10$	$-3.60+0$
2	$6.89+8$	$2.84+10$	$-3.51+0$
3	$1.68+8$	$3.60+10$	$-7.52+0$
4	$1.74+9$	$4.36+10$	$-5.61+0$
5	$6.24+7$	$5.04+10$	$-6.68+0$

$T \leq 10^{10}$ K and $\Omega^2 \sim \pi G\bar{\rho}$, the gravitational radiation is more important than radiation shear and bulk counterparts. While driving the r-modes to become unstable, it spins down the star. It is believed that the star loses much of its energy and angular momentum through the gravitational radiation in this stage. In a case study of Andersson & Kokkotas (2000), the rotational period increases from 2 ms to 19 ms in one year. Below $T = 10^8$ K and $\Omega^2 \sim \pi G\bar{\rho}/100$, the shear viscosity is the dominant factor in cooling down the star.

Acknowledgements. We wish to thank Sharon Morsink for a careful reading of the manuscript and illuminating comments. We would like to acknowledge the referee for his/her valuable comments.

Appendix A: The elements of S, C and W matrices

In calculating the elements of various matrices, the following parameters and integrals are encountered frequently:

$$Q_\ell = \left[\left(\ell^2 - m^2\right)/\left(2\ell-1\right)\left(2\ell+1\right)\right]^{1/2}, \tag{A.1}$$

$$\int \cos\vartheta Y^*_{km} Y_{\ell m} \mathrm{d}\Omega = Q_{\ell+1}\delta_{k,\ell+1} + Q_\ell \delta_{k,\ell-1}\,, \qquad \mathrm{d}\Omega = \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi, \tag{A.2}$$

$$\int \sin\vartheta Y^*_{km} \partial Y_{\ell m}/\partial\vartheta \mathrm{d}\Omega = \ell Q_{\ell+1}\delta_{k,\ell+1} - (\ell+1) Q_\ell \delta_{k,\ell-1}, \tag{A.3}$$

$$\int \left[\left(m^2/\sin^2\vartheta\right) Y^*_{km} Y_{\ell m} + \partial Y^*_{km}/\partial\vartheta \partial Y_{\ell m}/\partial\vartheta\right] \cos\vartheta \mathrm{d}\Omega = \ell(\ell+2) Q_{\ell+1}\delta_{k,\ell+1} + (\ell^2-1) Q_\ell \delta_{k,\ell-1}, \tag{A.4}$$

$$\int \cos^2\vartheta Y^*_{km} Y_{\ell m} \mathrm{d}\Omega = (Q^2_{\ell+1} + Q^2_\ell)\delta_{k\ell} + Q_{\ell-1}Q_\ell \delta_{k,\ell-2} + Q_{\ell+1}Q_{\ell+2}\delta_{k,\ell+2}, \tag{A.5}$$

$$\int \left[(m^2/\sin^2\vartheta) Y^*_{km} Y_{\ell m} + \partial Y^*_{km}/\partial\vartheta \partial Y_{\ell m}/\partial\vartheta\right] \times P_2(\cos\vartheta)\mathrm{d}\Omega = \frac{3}{2}\Big[\ell(\ell+3)Q^2_{\ell+1} + (\ell-2)(\ell+1)Q^2_\ell - \frac{1}{3}\ell(\ell+1)\Big]\delta_{k\ell} + \frac{3}{2}\left[(\ell-2)(\ell+1)Q_{\ell-1}Q_\ell\delta_{k,\ell-2} + \ell(\ell+3)Q_{\ell+1}Q_{\ell+2}\delta_{k,\ell+2}\right]. \tag{A.6}$$

The basic definitions for S, C, and W matrices are given in Eqs. (6). Expansions of ρ, p, and U entering these defining integrals are in Eqs. (2). The zero and Ω^2 orders of these variables are sufficient to carry out the analysis of Sect. 7 up to the Ω^4 order consistently.

Finally, the spherical harmonics form of the basis toroidal and poloidal vectors are given in Eqs. (10). Angular integrals entering the definition of any matrix element at any desired order are performed analytically. Integrals in radial directions are left for numerical calculations.

The S-matrix:

$$(S^{k\ell}_{0\rm tt})^{rs} = \delta_{k\ell}\ell(\ell+1)\int_0^R \rho_0 \phi^{r\ell}\phi^{s\ell}\mathrm{d}r, \tag{A.7}$$

$$(S^{k\ell}_{2\rm tt})^{rs} = \delta_{k\ell}\ell(\ell+1)\int_0^R \rho_{20}\phi^{r\ell}\phi^{s\ell}\mathrm{d}r + \frac{3}{2}\Big\{\delta_{k\ell}\left[\ell(\ell+3)Q^2_{\ell+1} + (\ell-2)(\ell+1)Q^2_\ell - \frac{1}{3}\ell(\ell+1)\right] + \delta_{k,\ell-2}(\ell-2)(\ell+1)Q_{\ell-1}Q_\ell + \delta_{k,\ell+2}\ell(\ell+3)Q_{\ell+1}Q_{\ell+2}\Big\} \times \int_0^R \rho_{22}\phi^{rk}\phi^{s\ell}\mathrm{d}r. \tag{A.8}$$

The C-matrix:

$$(C^{k\ell}_{0\rm tt})^{rs} = (m/\ell(\ell+1))(S^{k\ell}_{0\rm tt})^{rs}, \tag{A.9}$$

$$(C^{k\ell}_{2\rm tt})^{rs} = \delta_{k\ell} m \int_0^R \rho_{20}\phi^{r\ell}\phi^{s\ell}\mathrm{d}r + \frac{9}{2}m\big[\delta_{k\ell}\left(Q^2_{\ell+1} + Q^2_\ell - 1/9\right) + \delta_{k,\ell-2}Q_{\ell-1}Q_\ell + \delta_{k,\ell+2}Q_{\ell+1}Q_{\ell+2}\big]\int_0^R \rho_{22}\phi^{rk}\phi^{s\ell}\mathrm{d}r. \tag{A.10}$$

$$(C^{k\ell}_{0\rm pt})^{rs} = (C^{\ell k}_{0\rm tp})^{rs*} = -i\delta_{k,\ell+1}\ell Q_{\ell+1}\int_0^R \rho_0\left[\psi^{rk} + (\ell+2)\chi^{rk}\right]\phi^{s\ell}\mathrm{d}r + i\delta_{k,\ell-1}(\ell+1)Q_\ell\int_0^R \rho_0\left[\psi^{rk} - (\ell-1)\chi^{rk}\right]\phi^{s\ell}\mathrm{d}r. \tag{A.11}$$

The W-matrix:

$$(W_{0\mathrm{pp}}^{k\ell})^{rs} = \delta_{k\ell}\left\{\int_0^R \rho_0^{-1}\mathrm{d}p_0/\mathrm{d}\rho_0\delta_p^{r\ell}\rho_0(r)\delta_p^{s\ell}\rho_0(r)\mathrm{d}r/r^2 \right.$$
$$+\int_0^R [(\partial p_0/\partial\rho_0)_{\mathrm{ad}} - \mathrm{d}p_0/\mathrm{d}\rho_0]\rho_0(\mathrm{d}\psi^{rk}/\mathrm{d}r - \chi^{rk}/r)$$
$$\left.\times(\mathrm{d}\psi^{s\ell}/\mathrm{d}r - \chi^{s\ell}/r)\mathrm{d}r/r^2\right\}, \quad \text{(A.12)}$$

$$(W_{2\mathrm{pt}}^{k\ell})^{rs} = (W_{2\mathrm{tp}}^{\ell k})^{rk*}$$
$$= -3im\,[\delta_{k,\ell+1}Q_{\ell+1} + \delta_{k,\ell-1}Q_\ell]$$
$$\times\int_0^R \rho_0^{-1}\mathrm{d}p_0/\mathrm{d}\rho_0\rho_{22}\delta_p^{rk}\rho_0(r)\phi^{s\ell}\mathrm{d}r/r^2, \quad \text{(A.13)}$$

where

$$\delta_p^{r\ell}\rho_0(r) = \left[(1/\rho_0)(\mathrm{d}/\mathrm{d}r)(r\rho_0\psi^{r\ell}) - \ell(\ell+1)\chi^{r\ell}\right],$$

$$(W_{4\mathrm{tt}}^{k\ell})^{rs} = 9m^2[\delta_{k\ell}(Q_{\ell+1}^2 + Q_\ell^2) + \delta_{k,\ell-2}Q_{\ell-1}Q_\ell$$
$$+\delta_{k,\ell+2}Q_{\ell+1}Q_{\ell+2}]\int_0^R \rho_0^{-1}\mathrm{d}p_0/\mathrm{d}\rho_0\rho_{22}^2\phi^{rk}\phi^{s\ell}\mathrm{d}r/r^2. \quad \text{(A.14)}$$

Poloidal motions give rise to density perturbations and therefore to perturbations in self gravitation. In Eqs. (A.12) and (A.13) the latter is neglected (Cowling's approximation) for simplicity. Otherwise there is no conceptual difficulty in including another term in Eqs. (A.12)–(A.13) to avoid this approximation.

Finally, we note that the upper limit of all radial integrals here is the radius of the non-rotating star instead of that of the rotating one. We reproduce the argument of Sobouti (1980) to show that, in most cases, the effect arising from this difference in the limits of integration is of far higher order in Ω^2 than would influence the analysis of this paper. Let $\Delta R(\vartheta) = R(\vartheta) - R$ be the distance between two points with coordinate ϑ and situated on the surfaces of rotating and non-rotating stars. Obviously ΔR is of Ω^2 order. In a typical error integral, $\int_R^{R+\Delta R} f(r)\mathrm{d}r$, we expand $f(r)$ about R and obtain

$$\int_R^{R+\Delta R} f(r)\mathrm{d}r = f(R)\Delta R + \frac{1}{2}f'(R)(\Delta R)^2 + \cdots \quad \text{(A.15)}$$

In Eqs. (A.7)–(A.14) the integrands depend on a combination of the variables ρ_0, ρ_{20}, ρ_{22}, p_0 and p_{22}. For a star of effective polytropic index n, at the surface ρ_0 and p_0 vanish as $(R-r)^n$ and $(R-r)^{n+1}$. The leading error terms of Eq. (A.15) are $(\triangle R)^{n+1} \propto \Omega^{2(n+1)}$ and $(\triangle R)^{n+2} \propto \Omega^{2(n+2)}$, respectively. The distorted quantities ρ_{20}, ρ_{22} and p_{20}, p_{22} tend to zero as $(R-r)^{n+1}$ and $(R-r)^n$. However, considering the fact that these are second order quantities and should be multiplied by an extra factor of Ω^2 wherever they appear, the leading error term in expressions involving them are also of the order $\Omega^{2(n+1)}$ and $\Omega^{2(n+2)}$. Thus, the largest error committed in replacing the volume of the rotationally distorted star by that of the non-rotating one is of the order $\Omega^{2(n+1)}$, a matter of no concern for the analysis of this paper if $n \geq 1$.

Appendix B: Ansatz for the scalars $\phi^{s\ell}$, $\psi^{s\ell}$ and $\chi^{s\ell}$

The vicinity of the center of a star is a uniform medium, in the sense that, as r tends to zero, $\rho(r)$, $p(r)$, $U(r)$, etc. all tend to constant values. Any scalar function, $\sigma(r)$ say, associated with a wave in such a nondispersive uniform and isotropic environment should satisfy the wave equation $\nabla^2\sigma(r) + k^2\sigma(r) = 0$, k =const. Furthermore, if this scalar is associated with the spherical harmonic ℓ, i.e. if it is of form $\sigma(r)Y_\ell^m(\vartheta,\varphi)$ and is finite at the origin, should tend to zero as r^ℓ. Therefore σ should have an expansion of the form $\sigma(r) = r^\ell\sum_{s=0}^\infty a_s r^{2s}$. This is how the solutions of Laplace's equation ($k=0$), the spherical Bessel function and many other hypergeometric functions behave. A spherical harmonic vector, $\boldsymbol{\xi}^\ell$ belonging to ℓ, quite generally can be written in terms of three scalars

$$\boldsymbol{\xi}^\ell = -\boldsymbol{\nabla}(\sigma_1 Y_\ell^m) + \boldsymbol{\nabla}\times\boldsymbol{A}, \boldsymbol{A} = \hat{r}\sigma_2 Y_\ell^m + \boldsymbol{\nabla}\times(\boldsymbol{r}\sigma_3 Y_\ell^m), \quad \text{(B.1)}$$

where σ_i, $i = 1,2,3$ are scalars of the type described above. Therefore, the radial and non-radial components of $\boldsymbol{\xi}$ should have the form

$$\xi_r, \xi_\vartheta, \xi_\varphi \rightarrow r^\ell\sum_{s=1} b_s r^{2s-1}. \quad \text{(B.2)}$$

To ensure this behavior, it is sufficient that the scalars $\phi^{r\ell}$, $\psi^{r\ell}$, and $\chi^{r\ell}$ entering Eqs. (10) are proportional to $r^{\ell+2r}$.

We adopt the following ansatz for the

$$(\phi^{r\ell}, \psi^{r\ell}, \chi^{r\ell}) = \theta(r)r^{\ell+2r}, r = 0,1,2,\ldots \quad \text{(B.3)}$$

where $\theta(r)$ is the polytropic function, found by trial and error, that ensures faster convergence of the variational calculations. The ansatz has the required r^ℓ behavior at the center. Two remarks are in order here:

1) That a power set $\{r^{\ell+2r}, r = 0,1,2,\ldots\}$ is complete for expanding any function of r that behaves as r^ℓ near the origin follows from a theorem of Weieresstraus (Relich 1969; Dixit et al. 1979).

2) We have chosen the ansatz of Eq. (B.3) for their simplicity. They are not the most efficient ones for rapid convergence of variational calculations. The set of the asymptotic expressions that helioseismogists use for eigendisplacement vectors in the sun and other stars would, perhaps, give a faster convergence of the numerical computations, see Christensen-Dalsgaard (1998) and references therein.

Appendix C: Review of rotating polytropes

The structure of rotating polytropes is taken from a landmark paper of Chandrasekhar (1933). A summary of what is needed here with slight changes in his notation is as follows

$$\rho_0 = \rho_\mathrm{c}\theta^n, \rho_{20} = n\rho_\mathrm{c}\theta^{n-1}\Psi_0, \rho_{22} = n\rho_\mathrm{c}\theta^{n-1}\Psi_2, \quad \text{(C.1)}$$
$$p_0 = p_\mathrm{c}\theta^{n+1}, p_{20} = (n+1)p_\mathrm{c}\theta^n\Psi_0, p_{22} = (n+1)p_\mathrm{c}\theta^n\Psi_2, \quad \text{(C.2)}$$

where ρ_{c} and p_{c} are constants, θ is the polytropic variable and satisfies the Lane-Emden equation

$$(1/\eta^2)(\mathrm{d}/\mathrm{d}\eta)(\eta^2\mathrm{d}\theta/\mathrm{d}\eta) = -\theta^n, \qquad \eta = \left[(n+1)p_{\mathrm{c}}/4\pi G\rho_{\mathrm{c}}^2\right] r, \tag{C.3}$$

and Ψ_0, Ψ_2 satisfy the following

$$(1/\eta^2)(\mathrm{d}/\mathrm{d}\eta)(\eta^2\mathrm{d}\Psi_0/\mathrm{d}\eta) = -n\theta^{n-1}\Psi_0 + 1, \tag{C.4}$$

$$(1/\eta^2)(\mathrm{d}/\mathrm{d}\eta)(\eta^2\mathrm{d}\Psi_2/\mathrm{d}\eta) = (-n\theta^{n-1} + 6/\eta^2)\Psi_2. \tag{C.5}$$

It should be noted that Chandrasekhar's rotation parameter is $\Omega^2/2\pi G\rho_c$. For the purpose of this paper we have integrated Eqs. (C.3)–(C.5) numerically.

Appendix D: The energy of normal mode

Let us take the real part of Eq. (4) and write it in the following form

$$\mathcal{W}\boldsymbol{\xi}_{\mathrm{re}} + 2\rho\dot{\boldsymbol{\xi}}_{\mathrm{re}} \times \boldsymbol{\Omega} + \rho\ddot{\boldsymbol{\xi}}_{\mathrm{re}} = 0 \tag{D.1}$$

We take the scalar product of Eq. (D.1) by $\dot{\boldsymbol{\xi}}_{\mathrm{re}}$ and integrate over the volume of the star. The Coriolis term gives no contribution. The first term considering the hermitian character of $\mathcal{W}$ gives

$$\int \dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \mathcal{W}\boldsymbol{\xi}_{\mathrm{re}}\mathrm{d}^3x = \frac{1}{2}\int (\dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \mathcal{W}\boldsymbol{\xi}_{\mathrm{re}} + \boldsymbol{\xi}_{\mathrm{re}} \cdot \mathcal{W}\dot{\boldsymbol{\xi}}_{\mathrm{re}})\mathrm{d}^3x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{\xi}_{\mathrm{re}} \cdot \mathcal{W}\boldsymbol{\xi}_{\mathrm{re}}\mathrm{d}^3x. \tag{D.2}$$

The third term in Eq. (D.1) gives

$$\int \rho\dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \ddot{\boldsymbol{\xi}}_{\mathrm{re}}\mathrm{d}^3x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \dot{\boldsymbol{\xi}}_{\mathrm{re}}\mathrm{d}^3x. \tag{D.3}$$

Equations (D.1) and (D.2) should add to zero, which after a time integration gives the constant total energy

$$E = E_{\mathrm{kin}} + E_{\mathrm{pot}} = \frac{1}{2}\int (\rho\dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \dot{\boldsymbol{\xi}}_{\mathrm{re}} + \boldsymbol{\xi}_{\mathrm{re}} \cdot \mathcal{W}\boldsymbol{\xi}_{\mathrm{re}})\mathrm{d}^3x. \tag{D.4}$$

Next we substitute for $\mathcal{W}\boldsymbol{\xi}_{\mathrm{re}}$ from Eq. (D.1). After simple manipulations, we obtain

$$\begin{aligned} E &= \frac{1}{2}\int \rho(\dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \dot{\boldsymbol{\xi}}_{\mathrm{re}} - \boldsymbol{\xi}_{\mathrm{re}} \cdot \ddot{\boldsymbol{\xi}}_{\mathrm{re}})\mathrm{d}^3x \\ &= \frac{1}{2}\int \rho\left[2\dot{\boldsymbol{\xi}}_{\mathrm{re}} \cdot \dot{\boldsymbol{\xi}}_{\mathrm{re}} - \frac{\partial}{\partial t}(\boldsymbol{\xi}_{\mathrm{re}} \cdot \dot{\boldsymbol{\xi}}_{\mathrm{re}})\right]\mathrm{d}^3x \\ &= 2E_{\mathrm{kin}} - \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \rho\boldsymbol{\xi}_{\mathrm{re}} \cdot \boldsymbol{\xi}_{\mathrm{re}}\mathrm{d}^3x. \end{aligned} \tag{D.5}$$

Since the time dependence of $\boldsymbol{\xi}_{\mathrm{re}}$ is sinusoidal, upon taking the time average of Eq. (D.5) the second integral vanishes and we obtain

$$E = 2\overline{E_{\mathrm{kin}}} = E_{\mathrm{kin}}(\text{maximum}), \qquad \text{QED.} \tag{D.6}$$

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