Astronomy
&
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Dynamics of gravitational clustering

III. The quasi-linear regime for some non-Gaussian initial conditions

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Abstract. Using a non-perturbative method developed in a previous work (Paper II), we derive the probability distribution $\mathcal{P}(\delta_R)$ of the density contrast within spherical cells in the quasi-linear regime for some non-Gaussian initial conditions. We describe three such models. The first one is a straightforward generalization of the Gaussian scenario. It can be seen as a phenomenological description of a density field where the tails of the linear density contrast distribution would be of the form $\mathcal{P}_L(\delta_L) \sim e^{-|\delta_L|^{-\alpha}}$, where α is no longer restricted to 2 (as in the Gaussian case). We derive exact results for $\mathcal{P}(\delta_R)$ in the quasi-linear limit. The second model is a physically motivated isocurvature CDM scenario. Our approach needs to be adapted to this specific case and in order to get convenient analytical results we introduce a simple approximation (which is not related to the gravitational dynamics but to the initial conditions). Then, we find a good agreement with the available results from numerical simulations for the pdf of the linear density contrast for $\delta_{L,R} \gtrsim 0$. We can expect a similar accuracy for the non-linear pdf $\mathcal{P}(\delta_R)$. Finally, the third model corresponds to the small deviations from Gaussianity which arise in standard slow-roll inflation. We obtain exact results for the pdf of the density field in the quasi-linear limit, to first-order over the primordial deviations from Gaussianity.

Key words. cosmology: theory – large-scale structure of Universe

1. Introduction

In usual cosmological scenarios, large-scale structures in the universe are generated by the growth of small primordial density perturbations, through gravitational instability. At large scales or at early times one can use a perturbative approach to obtain the first few moments of the evolved density field. Moreover, as described in Bernardeau (1992) and Bernardeau (1994a) one can actually sum up the perturbative series at leading order in the limit $\sigma \to 0$ (where σ is the rms density fluctuation) to obtain all order cumulants. This yields the probability distribution function (pdf) $\mathcal{P}(\delta_R)$ of the density contrast within spherical cells. However, this method only applies to Gaussian initial conditions. Moreover, it may miss some non-perturbative effects.

In a previous paper (Paper II) we developed a non-perturbative method to derive the pdf $\mathcal{P}(\delta_R)$ in this quasi-linear regime. It is based on a steepest-descent approximation which yields asymptotically exact results in this limit. This allowed us to justify the results obtained by perturbative means and to correct some misconceptions

related to non-perturbative effects. Another advantage of our approach is that in principle it can also be applied to non-Gaussian primordial density fluctuations. However, the determination of the relevant saddle-points needed in order to get simple analytic results may then be more difficult. In particular, the method may need to be adapted to specific cases.

Thus, in this article we show in details how to obtain the pdf $\mathcal{P}(\delta_R)$ for three specific non-Gaussian models, in the quasi-linear regime. Indeed, although observations are consistent with Gaussian initial conditions so far, it is of interest to investigate a broader class of models until any non-Gaussianity is definitely ruled out. Moreover, there exist some reasonable physical scenarios (though somewhat more contrived than the standard CDM model) which give rise to non-Gaussian primordial fluctuations. Finally, even standard inflationary scenarios lead to small deviations from Gaussianity for the primordial density fluctuations.

This article is organized as follows. First, in Sect. 2 we recall the path-integral formulation which allows one to write an explicit expression for the pdf $\mathcal{P}(\delta_R)$ in terms of initial conditions. Then, in Sect. 3 we apply our method

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to a non-Gaussian model which is a straightforward generalization of the Gaussian case and could be used as a phenomenological tool to reproduce observations in case some departure from Gaussianity would be measured (e.g., in the CMB data). Then, in Sect. 4 we investigate a physically motivated model which describes an isocurvature cold dark matter scenario (Peebles 1999a). In particular, we obtain a good agreement with the available results from numerical simulations for the pdf of the linearly evolved density field. Finally, in Sect. 5 we consider the small primordial deviations from Gaussianity which arise in standard slow-roll inflation.

2. Generating functions. Gaussian initial conditions

In this article, we investigate the probability distribution function (pdf) $\mathcal{P}(\delta_R)$ of the density contrast δ_R within a spherical cell of comoving radius R, volume V:

$$\delta_R = \int_V \frac{\mathrm{d}^3 x}{V} \, \delta(\boldsymbol{x}). \tag{1}$$

Here $\delta(\boldsymbol{x})$ is the non-linear density contrast at the comoving coordinate $\boldsymbol{x}$, at the time of interest. In this section, following the method developed in Paper II, we recall how one can express the pdf $\mathcal{P}(\delta_R)$ as a function of the initial conditions through a path-integral formalism. In order to introduce our approach we first briefly consider the case of Gaussian initial conditions which is most familiar. In the next section, we extend our formalism to a non-Gaussian model which can be seen as the simplest generalization of the usual Gaussian case.

Rather than trying to directly evaluate the pdf itself, it is actually more convenient to study its Laplace transform $\psi(y)$ given by:

$$\psi(y) \equiv \langle \mathrm{e}^{-y\delta_R} \rangle \equiv \int_{-1}^{\infty} \mathrm{d}\delta_R \, \mathrm{e}^{-y\delta_R} \, \mathcal{P}(\delta_R). \tag{2}$$

Here, the symbol $\langle ... \rangle$ expresses the average over the initial conditions. Then, the last term in Eq. (2) can also be seen as the definition of the pdf $\mathcal{P}(\delta_R)$. Indeed, the pdf can be recovered from $\psi(y)$ through the standard inverse Laplace transform:

$$\mathcal{P}(\delta_R) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}y}{2\pi i} \, \mathrm{e}^{y\delta_R} \, \psi(y). \tag{3}$$

Moreover, the generating function $\psi(y)$ also yields the moments $\langle \delta_R^q \rangle$ through:

$$\psi(y) = \sum_{q=0}^{\infty} \frac{(-y)^q}{q!} \, \langle \delta_R^q \rangle. \tag{4}$$

Then, we need to compute $\psi(y)$ as an average over the initial conditions, using the first equality in Eq. (2). To do so, one simply needs two pieces of information. First, we must know the functional $\delta_R[\delta_{\mathrm{L}}(\boldsymbol{x})]$ which yields the

exact non-linear density contrast δ_R over the cell V which arises from the gravitational dynamics of the linear density field $\delta_{\mathrm{L}}(\boldsymbol{x})$. Indeed, as shown for instance in Paper I the initial conditions can be defined by the linear density contrast $\delta_{\mathrm{L}}(\boldsymbol{x})$ where we only keep the linear growing mode. This does not assume that the exact non-linear density field $\delta(\boldsymbol{x})$ can be written as a series expansion over powers of the linear field $\delta_{\mathrm{L}}(\boldsymbol{x})$. In fact, such a series is only asymptotic (e.g., Papers I, V). Hereafter, we note $\delta_{\mathrm{L}}(\boldsymbol{x})$ the growing mode of the linear density contrast at the time of interest, where we compute $\mathcal{P}(\delta_R)$ (i.e., to simplify notations we do not write explicitly the time dependence). Second, in order to perform the average in Eq. (2) we need the weight which is associated with all possible fields $\delta_{\mathrm{L}}(\boldsymbol{x})$. In other words, we must specify the probability distribution of the random field $\delta_{\mathrm{L}}(\boldsymbol{x})$. Thus, the first point expresses the physics of gravitational interactions while the second point describes the initial conditions of the system.

In the case of Gaussian initial conditions, the statistics of the random field $\delta_{\mathrm{L}}(\boldsymbol{x})$ are fully defined by the two-point correlation:

$$\Delta_{\mathrm{L}}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \langle \delta_{\mathrm{L}}(\boldsymbol{x}_1) \delta_{\mathrm{L}}(\boldsymbol{x}_2) \rangle. \tag{5}$$

The kernel Δ_{L} is symmetric, homogeneous and isotropic: $\Delta_{\mathrm{L}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Delta_{\mathrm{L}}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$. It is convenient to express Δ_{L} in terms of the power-spectrum $P(k)$ of the linear density fluctuations. To do so, we define the Fourier transform of the density field as:

$$\begin{cases} \delta(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{k} \, \mathrm{e}^{i\boldsymbol{k}.\boldsymbol{x}} \, \delta(\boldsymbol{k}) \\ \delta(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int \mathrm{d}\boldsymbol{x} \, \mathrm{e}^{-i\boldsymbol{k}.\boldsymbol{x}} \, \delta(\boldsymbol{x}). \end{cases} \tag{6}$$

Next, we define the Fourier transform of the kernel Δ_{L} by the property:

$$f_1.\Delta_{\mathrm{L}}.f_2 = \int \mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2 \, f_1(\boldsymbol{k}_1)^* . \Delta_{\mathrm{L}}(\boldsymbol{k}_1, \boldsymbol{k}_2) . f_2(\boldsymbol{k}_2) \tag{7}$$

for any real fields f_1 and f_2, where we introduced the short-hand notation:

$$f_1.\Delta_{\mathrm{L}}.f_2 \equiv \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2 \, f_1(\boldsymbol{x}_1).\Delta_{\mathrm{L}}(\boldsymbol{x}_1, \boldsymbol{x}_2).f_2(\boldsymbol{x}_2). \tag{8}$$

Using Eq. (6) this implies:

$$\Delta_{\mathrm{L}}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2 \, \mathrm{e}^{i(\boldsymbol{k}_2.\boldsymbol{x}_2 - \boldsymbol{k}_1.\boldsymbol{x}_1)} \, \Delta_{\mathrm{L}}(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{9}$$

which gives:

$$\Delta_{\mathrm{L}}(\boldsymbol{k}_1, \boldsymbol{k}_2) = (2\pi)^6 \, P(k_1) \, \delta_{\mathrm{D}}(\boldsymbol{k}_1 - \boldsymbol{k}_2) \tag{10}$$

where we defined the power-spectrum $P(k)$ of the linear density contrast by:

$$\langle \delta_{\mathrm{L}}(\boldsymbol{k}_1) \delta_{\mathrm{L}}(\boldsymbol{k}_2) \rangle \equiv P(k_1) \, \delta_{\mathrm{D}}(\boldsymbol{k}_1 + \boldsymbol{k}_2). \tag{11}$$

Finally, the inverse Δ_{L}^{-1} of the kernel Δ_{L} is (see Paper II):

$$\Delta_{\mathrm{L}}^{-1}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{1}{P(k_1)} \, \delta_{\mathrm{D}}(\boldsymbol{k}_1 - \boldsymbol{k}_2) \tag{12}$$

which implies that Δ_{L}^{-1} is positive definite since we have:

$$\delta_{\mathrm{L}}.\Delta_{\mathrm{L}}^{-1}.\delta_{\mathrm{L}} = \int \mathrm{d}\boldsymbol{k}\, \frac{|\delta_{\mathrm{L}}(\boldsymbol{k})|^2}{P(k)} \tag{13}$$

where we used $\delta_{\mathrm{L}}(-\boldsymbol{k}) = \delta_{\mathrm{L}}(\boldsymbol{k})^*$ for real fields $\delta_{\mathrm{L}}(\boldsymbol{x})$, see Eq. (6).

Thus, for Gaussian initial conditions the statistical properties of the random field $\delta_{\mathrm{L}}(\boldsymbol{x})$ are determined by the kernel Δ_{L} given in Eq. (10) (or equivalently by Δ_{L}^{-1} given in Eq. (12)). In particular, the average over the initial conditions in Eq. (2) can be written as the path-integral:

$$\psi(y) = \left(\mathrm{Det}\Delta_{\mathrm{L}}^{-1}\right)^{1/2} \int [\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})]\, \mathrm{e}^{-y\delta_R[\delta_{\mathrm{L}}]-\frac{1}{2}\delta_{\mathrm{L}}.\Delta_{\mathrm{L}}^{-1}.\delta_{\mathrm{L}}}. \tag{14}$$

This expression merely means that in order to compute the average $\langle \mathrm{e}^{-y\delta_R} \rangle$ we simply need to sum up the contributions of all possible linear density fields $\delta_{\mathrm{L}}(\boldsymbol{x})$ to which we associate a Gaussian weight which is proportional to $\mathrm{e}^{-(\delta_{\mathrm{L}}.\Delta_{\mathrm{L}}^{-1}.\delta_{\mathrm{L}})/2}$. The normalization factor $(\mathrm{Det}\Delta_{\mathrm{L}}^{-1})^{1/2}$ ensures that $\psi(0) = 1$, as implied by the definition (2). Here $\mathrm{Det}\Delta_{\mathrm{L}}^{-1}$ is the determinant of the kernel Δ_{L}^{-1}. Thus, Eq. (14) yields an explicit expression for the Laplace transform $\psi(y)$ (though it may be difficult to obtain numerical results from such a path-integral). This provides in turn the pdf $\mathcal{P}(\delta_R)$ through the inverse transform (3). In Paper II we showed how to get the generating function $\psi(y)$ from Eq. (14) in the limit $P(k) \to 0$ (i.e. for small rms density fluctuations) using a steepest-descent method. In the next section, we apply this method to a closely related non-Gaussian model.

3. Generalization of the Gaussian weight

In this section, we describe a non-Gaussian model which can be seen as the simplest generalization of the Gaussian scenario. This model is not derived from physical principles. It simply provides a phenomenological tool which can describe initial conditions such that the tails of the linear pdf $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L}})$ of the density field are of the form $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L}}) \sim \mathrm{e}^{-|\delta_{\mathrm{L}}|^{-\alpha}}$, where α is no longer restricted to 2 (as in the Gaussian case). In addition, it allows us to show on a simple example the power of the method we developed in Paper II and how it can be applied to non-Gaussian initial conditions.

3.1. A non-Gaussian model

In the case of Gaussian primordial density fluctuations, the average involved in the definition (2) of the generating function $\psi(y)$ (i.e. the Laplace transform of the pdf $\mathcal{P}(\delta_R)$) is simply given by the Gaussian weight $\exp(-\frac{1}{2}\delta_{\mathrm{L}}.\Delta_{\mathrm{L}}^{-1}.\delta_{\mathrm{L}})$ in the path-integral (14) which takes the average of $\mathrm{e}^{-y\delta_R}$ over all possible initial states, as we recalled in the previous section. A straightforward generalization of the Gaussian case (14) is to modify this weight. Thus, one

can consider the case where the average over the field δ_{L} is given by:

$$\psi(y) = \mathcal{N} \int [\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})]\, \mathrm{e}^{-y\delta_R-W[\delta_{\mathrm{L}}]} \tag{15}$$

with:

$$W[\delta_{\mathrm{L}}] = \sum_{i=1}^{N} \lambda_i\, (\delta_{\mathrm{L}}.\Delta_{\mathrm{W}}^{-1}.\delta_{\mathrm{L}})^{\alpha_i/2} \tag{16}$$

where $\alpha_1 < ... < \alpha_N$ and $\lambda_N > 0$. Here $\mathcal{N}$ is a normalization constant, so that $\psi(0) = 1$, and we still take the kernel Δ_{W}^{-1} to be of the form (12) hence Δ_{W}^{-1} is positive definite. We also define a quantity σ_{W} in terms of Δ_{W} by:

$$\sigma_{\mathrm{W}}(R)^2 \equiv \int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V} \frac{\mathrm{d}\boldsymbol{x}_2}{V}\, \Delta_{\mathrm{W}}(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{17}$$

However, contrary to the Gaussian case we now have:

$$\langle \delta_{\mathrm{L}}(\boldsymbol{x}_1)\delta_{\mathrm{L}}(\boldsymbol{x}_2) \rangle \neq \Delta_{\mathrm{W}}(\boldsymbol{x}_1, \boldsymbol{x}_2)\,, \quad \langle \delta_{\mathrm{L},R}^2 \rangle \neq \sigma_{\mathrm{W}}(R)^2. \tag{18}$$

The form (15) corresponds to a large density contrast tail of the form $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L},R}) \sim \mathrm{e}^{-|\delta_{\mathrm{L},R}|^{\alpha_N}}$ for the linear density field.

In order to compute the Laplace transform $\psi(y)$ we simply follow the steps of the calculation developed in Paper II. Thus, we first define a new generating function $\varphi(y)$ through:

$$\psi(y) = \mathrm{e}^{-\varphi(y\sigma_{\mathrm{W}}^{\alpha_N}, \sigma_{\mathrm{W}})/\sigma_{\mathrm{W}}^{\alpha_N}} \tag{19}$$

in order to factorize the amplitude of the kernel Δ_{W}. This yields from Eq. (15):

$$\mathrm{e}^{-\varphi(y,\sigma_{\mathrm{W}})/\sigma_{\mathrm{W}}^{\alpha_N}} = \mathcal{N} \int [\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})]\, \mathrm{e}^{-S_{\mathrm{W}}/\sigma_{\mathrm{W}}^{\alpha_N}} \tag{20}$$

where the "action" $S_{\mathrm{W}}[\delta_{\mathrm{L}}]$ is given by:

$$S_{\mathrm{W}}[\delta_{\mathrm{L}}] = y\, \delta_R[\delta_{\mathrm{L}}] + \sum_{i=1}^{N} \lambda_i\, \sigma_{\mathrm{W}}^{\alpha_N} (\delta_{\mathrm{L}}.\Delta_{\mathrm{W}}^{-1}.\delta_{\mathrm{L}})^{\alpha_i/2}. \tag{21}$$

The quasi-linear regime corresponds to the limit $\sigma_{\mathrm{W}} \to 0$ at fixed y. Then, we see that all terms with $1 \leq i \leq N-1$ in the action S_{W} in Eq. (21) vanish as $\propto \sigma_{\mathrm{W}}^{\alpha_N-\alpha_i}$ since $\Delta_{\mathrm{W}}^{-1} \propto \sigma_{\mathrm{W}}^{-2}$, see Eq. (17). Therefore, in the limit $\sigma_{\mathrm{W}} \to 0$ we are left with the action S_α:

$$S_\alpha[\delta_{\mathrm{L}}] = y\, \delta_R[\delta_{\mathrm{L}}] + \frac{\sigma_{\mathrm{W}}^\alpha}{\alpha}\, (\delta_{\mathrm{L}}.\Delta_{\mathrm{W}}^{-1}.\delta_{\mathrm{L}})^{\alpha/2} \tag{22}$$

with $\alpha = \alpha_N$ and we normalized Δ_{W} such that $\lambda_N = 1/\alpha$. The fact that the terms with $1 \leq i \leq N-1$ disappear in the quasi-linear limit expresses the fact that this limit is actually a "rare-event limit". Indeed, as seen in Paper II for the Gaussian case (and this remains valid here) a finite y corresponds to a finite density contrast. Then, in the limit $\sigma_{\mathrm{W}} \to 0$ these finite density contrasts (even though small) become very rare events. On the other hand, the tails of the functional distribution $\mathcal{P}[\delta_{\mathrm{L}}]$ defined by the weight $W[\delta_{\mathrm{L}}]$ are governed by the highest power in Eq. (16). This implies that in the quasi-linear limit the

only relevant term is the highest power α_N. Note that we can expect the validity of the quasi-linear limit to extend to larger values of σ_W in the case $N = 1$ than for $N > 1$. Indeed, in this latter case we can expect the terms $1 \leq i \leq N-1$ to make a significant contribution for $\sigma_W \sim 1$ which is not taken into account at all by the quasi-linear limit.

3.2. Steepest-descent method

The action S_α is independent of the normalization of the kernel Δ_W. Thus, as in Paper II we can apply the steepest-descent method in the limit $\sigma_W \to 0$. Indeed, it is clear than in this limit the path-integral (20) is dominated by the global minimum of the action $S_\alpha[\delta_L]$ while the contributions from other points δ_L are exponentially damped. Then, the steepest-descent method yields asymptotically exact results in this limit. We briefly describe below the main steps of this steepest-descent method. Since the derivation is very close to the one performed in Paper II for the Gaussian case we refer the reader to that paper for the details of the calculation. Note that the Gaussian case actually corresponds to $\alpha = 2$.

In order to apply the steepest-descent approximation we first need to find the global minimum of the action $S_\alpha[\delta_L]$. The condition which expresses that the point δ_L is an extremum (or a saddle-point) is:

$$\frac{\delta S_\alpha}{\delta(\delta_L(\boldsymbol{x}))} = 0 \quad \text{for all} \quad \boldsymbol{x}, \tag{23}$$

where $\delta/\delta(\delta_L(\boldsymbol{x}))$ is the functional derivative with respect to δ_L at the point $\boldsymbol{x}$. This constraint also writes:

$$\delta_L(\boldsymbol{x}') = \frac{-y}{\sigma_W^\alpha} (\delta_L.\Delta_W^{-1}.\delta_L)^{1-\alpha/2}$$
$$\times \int \mathrm{d}\boldsymbol{x}'' \, \Delta_W(\boldsymbol{x}', \boldsymbol{x}'') \, \frac{\delta(\delta_R)}{\delta(\delta_L(\boldsymbol{x}''))}. \tag{24}$$

Thus, we obtain exactly the same profile for the saddle-point as in the Gaussian case. Indeed, we can see that $\alpha \neq 2$ only modifies a numerical multiplicative factor in the r.h.s. of Eq. (24) (i.e. independent of $\boldsymbol{x}'$ and $\boldsymbol{x}''$) which could formally be absorbed into y through $y \to y\sigma_W^{2-\alpha}(\delta_L.\Delta_W^{-1}.\delta_L)^{1-\alpha/2}$. Then, as in Paper II we obtain a spherically symmetric saddle-point. Note that the existence of a spherical saddle-point could be expected a priori, since the initial conditions are homogeneous and isotropic and we study the density contrast within spherical cells. Therefore, the very problem we investigate is spherically symmetric. However, a priori this spherical saddle-point is not necessarily the global minimum of the action (in full generality it might as well be a maximum). This will need to be checked a posteriori.

The fact that we obtain a spherical saddle-point greatly simplifies the problem since it means that this point δ_L is described by the well-known spherical collapse solution of the gravitational dynamics. This reads:

$$\begin{cases} \delta_R = \mathcal{F}[\delta_{L,R_L}] \\ R_L^3 = (1 + \delta_R)R^3 \end{cases} \tag{25}$$

where the function $\mathcal{F}[\delta_{L,R_L}]$ is given by the usual spherical collapse solution of the equations of motion (e.g., Peebles 1980; Paper II). The second equation in (25) merely expresses the conservation of mass and the fact that the matter enclosed within the radius R in the actual non-linear density field actually comes from a Lagrangian co-moving radius R_L, for a spherical initial condition. The Eq. (25) provides the functional $\delta_R[\delta_L(\boldsymbol{x})]$ for spherical states $\delta_L(\boldsymbol{x})$. As shown in Paper II this is sufficient to derive the spherical saddle-point which obeys the condition (24). Using the results of Paper II with the substitution $y \to y\sigma_W^{2-\alpha}(\delta_L.\Delta_W^{-1}.\delta_L)^{1-\alpha/2}$ we eventually obtain:

$$\delta_{L,R_L} = -\frac{y}{\sigma_W^{\alpha-2}(\delta_L.\Delta_W^{-1}.\delta_L)^{(\alpha-2)/2}}$$
$$\times \frac{\mathcal{F}'[\delta_{L,R_L}]\sigma_W^2(R_L)/\sigma_W^2(R)}{1 - \mathcal{F}'[\delta_{L,R_L}]\frac{R^3}{3R_L^2}\delta_{L,R_L}\frac{1}{\sigma_W(R_L)}\frac{\mathrm{d}\sigma_W}{\mathrm{d}R}(R_L)} \tag{26}$$

together with:

$$\delta_{L,R'} = \delta_{L,R_L} \frac{\Delta_W(R', R_L)}{\Delta_W(R_L, R_L)}. \tag{27}$$

These two equations fully define the spherically symmetric saddle-point $\delta_L(\boldsymbol{x})$. The implicit Eq. (26) determines δ_{L,R_L} while the radial profile of this initial state is given by Eq. (27). In Eq. (27) we introduced the spherically averaged kernel $\Delta_W(R_1, R_2)$ defined by:

$$\Delta_W(R_1, R_2) \equiv \int_{V_1} \frac{\mathrm{d}\boldsymbol{x}_1}{V_1} \int_{V_2} \frac{\mathrm{d}\boldsymbol{x}_2}{V_2} \, \Delta_W(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{28}$$

As in Paper II these equations can be simplified by introducing the functions $\tau(\delta_L)$ and $\mathcal{G}(\tau)$ defined by:

$$\tau(\delta_L) \equiv \frac{-\delta_L\,\sigma_W(R)}{\sigma_W\left[(1 + \mathcal{F}[\delta_L])^{1/3}R\right]} \tag{29}$$

and:

$$\mathcal{G}(\tau) \equiv \mathcal{F}[\delta_L(\tau)] = \delta_R \tag{30}$$

where $\delta_L(\tau)$ is defined by Eq. (29). Using Eq. (29), we note that for a power-law "power-spectrum" $P(k) \propto k^n$ we have $\sigma_W(R) \propto R^{-(n+3)/2}$ so that Eq. (30) simplifies to:

$$\mathcal{G}(\tau) = \mathcal{F}\left[-\tau\,(1 + \mathcal{G}[\tau])^{-(n+3)/6}\right]. \tag{31}$$

Note however that in this non-Gaussian case the "power-spectrum" $P(k)$ does not obey Eq. (11). Nevertheless, it still provides a measure of the power associated with the wavenumber k.

Finally, the limiting generating function $\varphi(y)$ defined by $\varphi(y) \equiv \varphi(y, \sigma = 0)$ is given by the value of the action S_α at this minimum which yields the implicit system:

$$\begin{cases} \tau |\tau|^{\alpha-2} = -y \, \mathcal{G}'(\tau) \\[2mm] \varphi(y) = y \, \mathcal{G}(\tau) + \dfrac{|\tau|^\alpha}{\alpha}. \end{cases} \tag{32}$$

Of course, this result is similar to the Gaussian case which can be recovered by taking $\alpha = 2$. Note that Eq. (32) has only been derived for real y (hence τ is real), so that $|\tau|$ is the absolute value of τ.

Thus, the steepest-descent method allows us to compute the Laplace transform $\psi(y)$ in the quasi-linear regime through Eq. (32) which yields $\psi(y)$ as in Eq. (19). This will give the pdf $\mathcal{P}(\delta_R)$ through Eq. (3). However, we first need to check that the spherical saddle-point defined by Eq. (26) and Eq. (27) is the global minimum of the action S_α. For small positive y this can be rigorously proved as for the Gaussian case, using the fact that the kernel Δ_{L}^{-1} is positive definite. We shall not repeat this discussion here (see Sect. 3.4 in Paper II). For negative y this cannot be proved as it may actually happen that the spherical saddle-point is only a local minimum. This also occurs in the Gaussian case for a power-spectrum with $n < 0$. This case is discussed in great details in Paper II where we show that the steepest-descent method remains useful but requires a careful justification. Since the same discussion can be applied to the non-Gaussian model we investigate here we refer the reader to Paper II for a description of such cases and we shall only give a brief comment on this point in the next section.

3.3. Geometrical construction

As for the Gaussian case studied in Paper II we can give a geometrical construction of the generating function $\varphi(y)$ defined by Eq. (32). Indeed, this latter expression can also be written:

$$\varphi(y) = \min_\tau \left[y \, \mathcal{G}(\tau) + \frac{|\tau|^\alpha}{\alpha} \right]. \tag{33}$$

The minimum which appears in this expression merely expresses the fact that by sheer definition of the steepest-descent method the generating function $\varphi(y)$ is governed by the global minimum of the action S_α.

Then, we see that the geometrical construction displayed in Fig. 4 in Paper II still applies to this non-Gaussian case, with the modification that the parabola $h - \tau^2/(2y)$ are replaced by the curves $h - |\tau|^\alpha/(\alpha y)$. That is, the minimum point τ which yields $\varphi(y)$ in Eq. (33) is given by the first contact of the curve $\mathcal{G}(\tau)$ with the curves $h - |\tau|^\alpha/(\alpha y)$ of varying height h. For $y > 0$ we start from below at $h = -\infty$ while for $y < 0$ we start from above at $h = +\infty$.

As in the Gaussian case, we find that for some values of n the action S_α has no global minimum. This can be seen from the geometrical construction shown in Fig. 4 or

simply from Eq. (33). Indeed, as noticed in Paper II from Eq. (31) we have the following asymptotic behaviour for $\mathcal{G}(\tau)$:

high densities: $\tau \to -\infty$, $\mathcal{G} \to \infty$: $\mathcal{G} \sim (-\tau)^{6/(n+3)}$. (34)

Using this expression we see that for $y < 0$ the term in the brackets in the r.h.s. of Eq. (33) is not bounded from below if $\alpha < 6/(3 + n)$, since in this case it goes to $-\infty$ for $\tau \to -\infty$. For the Gaussian case this occurs for $n < 0$. However, as explained in Paper II the steepest-descent method developed in Sect. 3.2 remains useful and the pdf $\mathcal{P}(\delta_R)$ is still governed by the spherical saddle-point defined by Eqs. (26) and (27). This can be seen by the following remark. If we consider the quantity $\delta_R^{1/q}$ rather than δ_R, where q is an odd integer, we can again apply the steepest-descent method which yields the associated generating function $\varphi_q(y)$ as in Eq. (33). However, the function $\mathcal{G}_q(\tau)$ is now given by $\mathcal{G}_q(\tau) = \mathcal{G}(\tau)^{1/q}$. Then, if we choose a sufficiently large value for q the behaviour of the r.h.s. bracket in Eq. (33) is dominated by the term $|\tau|^\alpha$ at large $|\tau|$ so that the action now exhibits a global minimum. Of course, this point is still the spherical saddle-point we obtained in Sect. 3.2 (since the initial conditions δ_{L} which give rise to a fixed value of δ_R do not depend on whether we study δ_R itself or $\delta_R^{1/q}$!). Then the pdf $\mathcal{P}(\delta_R)$ could be recovered from $\mathcal{P}(\delta_R^{1/q})$ through a mere change of variable. This implies in turn that the pdf $\mathcal{P}(\delta_R)$ is still governed by the standard spherical saddle-point, even though the latter is no longer the global minimum of the action. However, as described in Paper II the treatment of this case requires some additional care and it is actually associated with a break-up of perturbative theory.

3.4. Calculation of the pdf $\mathcal{P}\delta_R$

Finally, we describe how to compute the pdf $\mathcal{P}(\delta_R)$ itself from the results obtained in Sect. 3.2. We consider the case $n = -1$ and $\alpha = 3$ where the action S_α always shows a global minimum. Then, the Laplace transform $\psi(y)$ is fully defined by the generating function $\varphi(y)$ given by Eq. (32).

In order to compute the pdf from Eq. (32) we simply use the inverse relation (3) which yields:

$$\mathcal{P}(\delta_R) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}y}{2\pi i \sigma_{\mathrm{W}}^\alpha} \, \mathrm{e}^{[y\delta_R - \varphi(y)]/\sigma_{\mathrm{W}}^\alpha}. \tag{35}$$

Then, to perform the integration over y we need the analytic continuation of $\varphi(y)$ over the complex plane. Indeed, note that Eqs. (32) and (33) were actually derived for real y (hence τ was real). In particular, we must continue the absolute value $|\tau|$ over the complex plane. Moreover, we need to specify the integration path over y (since $\varphi(y)$ will usually be singular at the origin). First, the integration path starts from the real axis at the saddle-point $(\tau_{\mathrm{c}}, y_{\mathrm{c}})$ given by:

$$\frac{\mathrm{d}\chi}{\mathrm{d}y}(y_{\mathrm{c}}) = 0 \quad \text{with} \quad \chi(y) \equiv \delta_R y - \varphi(y). \tag{36}$$

From Eq. (32) we have:

$$\mathcal{G}(\tau_c) = \delta_R \text{ since } \varphi'(y) = \mathcal{G}(\tau). \qquad (37)$$

Thus, as for the Gaussian case, we see from Eq. (30) that the triplet (δ_R, τ_c, y_c) is also the triplet (δ_R, τ, y) we obtained in Sect. 3.2 to get $\varphi(y)$. This simply means that $\mathcal{P}(\delta_R)$ at the point δ_R is governed by the neighbourhood of the saddle-point $\delta_L(\boldsymbol{x})$ obtained in Sect. 3.2, which obeys $\mathcal{F}(\delta_{L,R_L}) = \delta_R$. This is actually quite natural. Then, we need to continue the function $\varphi(y)$ over the complex plane from the neighbourhood of the point y_c on the real axis. Thus, if $\tau_c > 0$ (i.e. $y_c > 0$) we replace $|\tau|$ in Eq. (32) by τ, while if $\tau_c < 0$ (i.e. $y_c < 0$) we replace $|\tau|$ by $(-\tau)$. Then, we have the usual analytic continuation of the power-laws τ^α or $(-\tau)^\alpha$. The argument of these power-laws is real positive at the starting point (τ_c, y_c). This provides the analytic continuation of the generating function $\varphi(y)$. The function $\mathcal{G}(\tau)$ is also analytic around the real axis because the spherical collapse solution $\mathcal{F}(\delta_L)$ can be expressed in terms of analytic functions (e.g., trigonometric or hyperbolic functions for a critical density universe).

Next, the integration path in the complex plane follows the steepest-descent contour which runs through the saddle-point y_c. It is given by the constraint $\text{Im}(\chi) = 0$ and it is orthogonal to the real axis at the point y_c. This path is also symmetric about the real axis which implies that the result for $\mathcal{P}(\delta_R)$ is real. Note that in the quasi-linear limit $\sigma_W \to 0$ the contributions to the integral (35) only come from an infinitesimal neighbourhood of the saddle-point (τ_c, y_c) around the real axis.

Usually the function $\varphi(y)$ is not regular at the origin since we have $\varphi(y) \simeq -(1 - 1/\alpha)|y|^{\alpha/(\alpha-1)}$ for small real y. Then, the pdf $\mathcal{P}(\delta_R)$ is not analytic at $\delta_R = 0$ and the moments of the pdf cannot be recovered from the expansion (4). This is not the case for Gaussian initial conditions where both $\varphi(y)$ and $\mathcal{P}(\delta_R)$ are regular at the origin.

As an example, we show in Fig. 1 the pdf $\mathcal{P}(\delta_R)$ for the case $\alpha = 3$ and $n = -1$ (solid line). We also have $\sigma = 0.74$ where $\sigma^2 \equiv \langle \delta_{L,R}^2 \rangle$ is the rms density fluctuation. Since $\sigma \neq \sigma_W$ we numerically compute σ from the second moment of the linear pdf $\mathcal{P}_L$. Indeed, the steepest-descent method developed in Sect. 3.2 also yields the linear pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly-evolved density field. In this case we have $\delta_R = \delta_L$ and $R_L = R$, hence we simply need to take $\mathcal{G}(\tau) = -\tau$ in Eq. (32) (of course, for $\alpha = 2$ this gives back the Gaussian). The non-linear evolution of the density field increases the value of $\mathcal{P}(\delta_R)$ at large positive δ_R (it actually spreads the linear pdf towards larger value of the density) as in the Gaussian case but the cutoff remains much sharper. It also induces a cutoff at low densities $\delta_R \geq -1$ which expresses the fact that the actual density ρ is positive.

For comparison, we also display in Fig. 1 the results we obtain for Gaussian initial conditions with the same linear variance σ (dotted lines). These curves were derived in Paper II. As could be anticipated from Eq. (15) the large density contrast cutoff of the linear pdf is much sharper than for the Gaussian case: it goes as $\mathcal{P}(\delta_{L,R}) \sim e^{-|\delta_{L,R}|^3}$
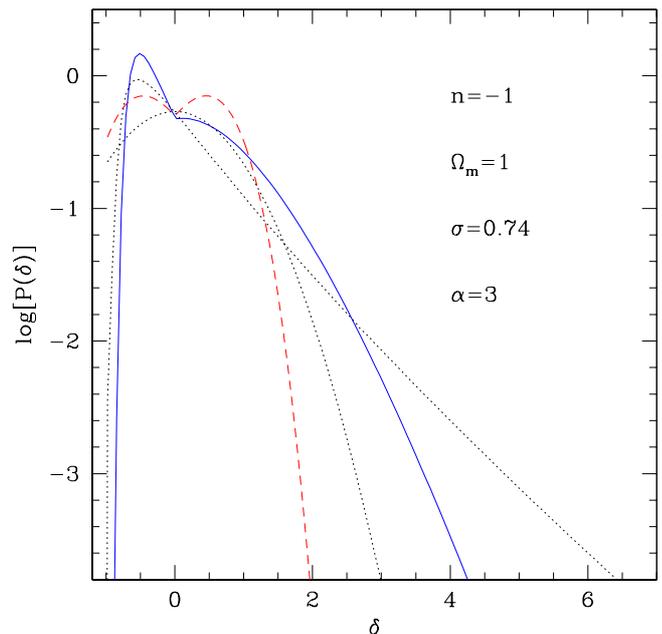
Fig. 1. The pdf $\mathcal{P}(\delta_R)$ for $n = -1$, $\Omega_m = 1$, $\sigma = 0.74$ and $\alpha = 3$. This corresponds to $\sigma_W = 0.65$. The solid line shows the theoretical prediction from (32). The dashed-line displays the linear pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly evolved density field. The dotted-lines show the linear and non-linear pdfs obtained for Gaussian initial conditions with the same linear rms density fluctuation σ (the non-linear pdf is the asymmetric curve with the extended high-density tail).

since here we take $\alpha = 3$. Moreover, we can check that for the actual non-linear pdfs this trend remains valid. More precisely, as shown in Paper II the high-density tail of the non-linear pdf for Gaussian initial conditions is of the form:

Gaussian scenario: $\delta_R \gg 1$: $\mathcal{P}(\delta_R) \sim e^{-\delta_R^{(n+3)/3}/\sigma^2}$ (38)

while for non-Gaussian initial conditions we obtain:

non-Gaussian: $\delta_R \gg 1$: $\mathcal{P}(\delta_R) \sim e^{-\delta_R^{\alpha(n+3)/6}/\sigma_W^\alpha}$. (39)

These behaviours are obtained from Eqs. (32), (35) and (37) which yield:

$$\mathcal{P}(\delta_R) \sim e^{-|\tau_c|^\alpha/(\alpha\sigma_W^\alpha)} \qquad (40)$$

while the asymptotic behaviour of $\tau_c(\delta_R)$ for large δ_R is given by Eqs. (34) and (37).

As noticed above, the non-Gaussian model (16) is not derived from a physical scenario of the primordial universe. It should be viewed as a simple model to describe initial density fluctuations which exhibit a non-Gaussian tail. Then, the analysis performed in the previous sections shows how one may derive in a rigorous manner the gravitational dynamics of density fluctuations in the quasi-linear regime. Moreover, we can expect the form (39) of the high-density tail to apply to any model which obeys $\mathcal{P}(\delta_{L,R}) \sim e^{-|\delta_{L,R}|^3}$ at large densities, whatever the details of the model. In fact, we have shown that in the

quasi-linear regime the tail (39) (actually the generating function $\varphi(y)$ itself !) is common to all models of the class described by Eq. (16) with $\alpha_N = \alpha$.

In practice, since observations have not shown any deviations from Gaussianity so far, the parameter α is constrained to be close to 2 (the Gaussian value). Note that this family of models presents the advantage of containing the usual Gaussian scenario as a particular case.

4. An isocurvature CDM model

In addition to generalized non-Gaussian models as the one studied in Sect. 3 which are rather ad-hoc tools, there exist some specific non-Gaussian models which arise from a physically motivated description of the primordial universe. These scenarios are of great interest as they could provide a reasonable alternative to the standard simple inflationary model. Moreover, it is prudent not to disregard a priori sensible models of structure formation in view of the rather indirect character of the probes of the early universe which are available to us.

4.1. Initial conditions

Thus, in this section we consider a specific isocurvature cold dark matter model which was presented in Peebles (1999a). This is an inflationary scenario which involves three scalar fields and it gives birth to non-Gaussian isocurvature CDM fluctuations. We refer the reader to Peebles (1999a) for a description of the physical processes which give rise to this scenario. Then, at the end of inflation (at time t_i) the CDM mass distribution is (Peebles 1999a):

$$\rho_i(\boldsymbol{x}) \propto \phi_i(\boldsymbol{x})^2, \qquad (41)$$

where $\phi_i(\boldsymbol{x})$ is a Gaussian random field with zero mean and power-spectrum P_{ϕ_i}:

$$\langle \phi_i(\boldsymbol{k}_1)\phi_i(\boldsymbol{k}_2) \rangle \equiv P_{\phi_i}(k_1)\delta_{\mathrm{D}}(\boldsymbol{k}_1 + \boldsymbol{k}_2). \qquad (42)$$

On the scales of interest for structure formation we have:

$$P_{\phi_i}(k) \propto k^{n_\phi} \;,\; n_\phi \simeq -2.4 \qquad (43)$$

while below the coherence length x_c (i.e. for $k \gg 1/x_c$) the power-spectrum decreases as $P_{\phi_i}(k) \propto k^s$ with $s < -3$. This coherence length ensures that $\langle \phi_i(\boldsymbol{x})^2 \rangle$ is finite, as required by Eq. (41). In the case of the model described in Peebles (1999a) we have $x_c \simeq 10$ pc. In the linear regime below the horizon the CDM density perturbations grow as $\delta_{\mathrm{L}}(\boldsymbol{x}, t) \propto D_+(t)$ (we only keep the growing mode) hence we write:

$$\delta_{\mathrm{L}}(\boldsymbol{x}, t) = D_+(t) \left(\phi_i(\boldsymbol{x})^2 - 1 \right) \qquad (44)$$

where we normalized $\phi_i(\boldsymbol{x})$ by $\langle \phi_i(\boldsymbol{x})^2 \rangle = 1$ and the growing mode by $D_+(t_i) = 1$. Note that in Eq. (44) and in the following we only consider "small" comoving

scales $x \lesssim 14$ Mpc, where the isocurvature CDM transfer function is close to unity (Peebles 1999b). On larger scales the density fluctuations are damped with respect to Eq. (44) and we should take into account the decrease of the transfer function. Since we are interested in the statistics of the density field at an arbitrary time t it is convenient to rescale the Gaussian field ϕ_i. Thus, we define $\phi(\boldsymbol{x}) \equiv D_+(t)^{1/2}\phi_i(\boldsymbol{x})$ so that Eq. (44) now reads:

$$\delta_{\mathrm{L}}(\boldsymbol{x}) = \phi(\boldsymbol{x})^2 - \langle \phi^2 \rangle \qquad (45)$$

at the redshift of interest. The power-spectrum $P_\phi(k)$ is still of the form (43). Note that the relation between the fields $\delta_{\mathrm{L}}(\boldsymbol{x}, t)$ and $\phi(\boldsymbol{x}, t)$ depends explicitly on time through the last term in Eq. (45) (to simplify notations we usually do not write explicitly the time dependence of the various random fields). Indeed, we must recall here that $\langle \phi^2 \rangle$ is not the spatial average of $\phi(\boldsymbol{x})^2$ but the mean over all realizations of the Gaussian random field ϕ. Therefore, it is independent of the peculiar realization ϕ which gives rise to a given linear density field through Eq. (45) (even though by ergodicity both averages happen to be equal). We define the two-point correlation of the field $\phi(\boldsymbol{x})$ by:

$$\Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2) \rangle \qquad (46)$$

and the linear variance over the scale R by:

$$\sigma_\phi^2(R) \equiv \int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V} \frac{\mathrm{d}\boldsymbol{x}_2}{V} \, \Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2). \qquad (47)$$

Note that Eq. (46) also implies that we can define the average $\langle \phi^2 \rangle$ by:

$$\langle \phi^2 \rangle \equiv \Delta_\phi(\boldsymbol{x}, \boldsymbol{x}) \qquad (48)$$

which does not depend on $\boldsymbol{x}$ since the initial conditions are invariant through translations. From Eq. (45) and the fact that $\phi(\boldsymbol{x})$ is Gaussian one obtains the variance $\sigma(R)$ of the linearly-evolved density contrast as:

$$\sigma^2(R) \equiv \langle \delta_{\mathrm{L},R}^2 \rangle = 2 \int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V} \frac{\mathrm{d}\boldsymbol{x}_2}{V} \, \Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2)^2 \qquad (49)$$

since the two-point correlation Δ_{L} of the linear density field is given by:

$$\Delta_{\mathrm{L}}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \langle \delta_{\mathrm{L}}(\boldsymbol{x}_1)\delta_{\mathrm{L}}(\boldsymbol{x}_2) \rangle = 2\Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2)^2. \qquad (50)$$

From Eqs. (47) and (49) we see that for $-3 < n_\phi < -3/2$ we get $\sigma(R) \propto \sigma_\phi^2(R)$ and the slope n of the linear power-spectrum $P(k)$ of the density fluctuations is related to n_ϕ by: $n = 3 + 2n_\phi$. This implies $n_\phi \simeq -2.4$ as in Eq. (43) in order to match observational constraints (Peebles 1999b). For numerical calculations we shall use:

$$n_\phi = -2.4 \quad \text{and} \quad n = 3 + 2n_\phi = -1.8. \qquad (51)$$

4.2. Laplace transform

We again define the generating function $\psi(y)$ as in Eq. (2) but the Gaussian average over $\delta_L(\boldsymbol{x})$ which appeared in Eq. (14) must now be replaced by a Gaussian average over $\phi(\boldsymbol{x})$:

$$\psi(y) = \mathcal{N} \int [\mathrm{d}\phi(\boldsymbol{x})] \ \mathrm{e}^{-y\delta_R[\phi] - \frac{1}{2}\phi.\Delta_\phi^{-1}.\phi}. \tag{52}$$

Thus, our goal is now to estimate the Laplace transform $\psi(y)$. To do so, we shall not introduce a rescaled generating function $\varphi(y)$ as in Eq. (19) and we work directly with $\psi(y)$. First, using Eq. (45) it is convenient to introduce the linear density field $\delta_L(\boldsymbol{x})$ into the expression (52) through the Dirac functional $\delta_D[\langle\phi^2\rangle + \delta_L(\boldsymbol{x}) - \phi(\boldsymbol{x})^2]$:

$$\psi(y) = \int [\mathrm{d}\phi(\boldsymbol{x})][\mathrm{d}\delta_L(\boldsymbol{x})] \ \delta_D[\langle\phi^2\rangle + \delta_L(\boldsymbol{x}) - \phi(\boldsymbol{x})^2]$$
$$\times \ \mathrm{e}^{-y\delta_R[\delta_L] - \frac{1}{2}\phi.\Delta_\phi^{-1}.\phi}. \tag{53}$$

Here and in the following we do not write the normalization constant of the path-integral (it will be recovered in the final expressions). Then, using the integral representation of the Dirac functional we obtain:

$$\psi(y) = \int [\mathrm{d}\phi(\boldsymbol{x})][\mathrm{d}\delta_L(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})]$$
$$\times \ \mathrm{e}^{-y\delta_R[\delta_L] + i\lambda.(\langle\phi^2\rangle + \delta_L - \phi^2) - \frac{1}{2}\phi.\Delta_\phi^{-1}.\phi} \tag{54}$$

where $\lambda(\boldsymbol{x})$ is an auxiliary real field. Note that the exponent is actually quadratic over ϕ and we can write Eq. (54) as:

$$\psi(y) = \int [\mathrm{d}\phi(\boldsymbol{x})][\mathrm{d}\delta_L(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})]$$
$$\times \ \mathrm{e}^{-y\delta_R[\delta_L] + i\lambda.(\langle\phi^2\rangle + \delta_L) - \frac{1}{2}\phi.(\Delta_\phi^{-1} + 2i\Lambda).\phi} \tag{55}$$

where we introduced the kernel:

$$\Lambda(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \lambda(\boldsymbol{x}_1)\delta_D(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{56}$$

Then, the Gaussian integration over $\phi(\boldsymbol{x})$ is straightforward and it yields:

$$\psi(y) = \int [\mathrm{d}\delta_L(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})] \left[\mathrm{Det}(\Delta_\phi^{-1} + 2i\Lambda)\right]^{-1/2}$$
$$\times \ \mathrm{e}^{-y\delta_R + i\lambda.(\langle\phi^2\rangle + \delta_L)}. \tag{57}$$

Note indeed that the factor $\langle\phi^2\rangle$ in Eq. (55) does not depend on the random field $\phi(\boldsymbol{x})$ over which we integrate, as shown by Eq. (48). Next, multiplying the r.h.s. of Eq. (57) by $(\mathrm{Det}\Delta_\phi)^{-1/2}$ (i.e. we change the implicit normalization constant of the path-integral) and using the relation $\ln(\mathrm{Det}M) = \mathrm{Tr}(\ln M)$ we get:

$$\psi(y) = \int [\mathrm{d}\delta_L(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})] \ \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\ln(1+2i\Lambda\Delta_\phi)}$$
$$\times \ \mathrm{e}^{-y\delta_R + i\lambda.(\langle\phi^2\rangle + \delta_L)}. \tag{58}$$

At this point, it is instructive to consider the linear regime when the density field is linearly evolved: $\delta(\boldsymbol{x}, t) = \delta_L(\boldsymbol{x}, t) \propto D_+(t)$. In this case, we have:

$$y \ \delta_R = y \int \mathrm{d}\boldsymbol{x} \ \frac{\theta(x < R)}{V} \ \delta_L(\boldsymbol{x}) \tag{59}$$

where $\theta(x < R)$ is a top-hat with obvious notations. Then, the integration over $\delta_L(\boldsymbol{x})$ is straightforward and it yields the Dirac functional $\delta_D[\lambda(\boldsymbol{x}) + iy\theta(x < R)/V]$. The integration over $\lambda(\boldsymbol{x})$ is now obvious and we obtain for the generating function $\psi_L(y)$ which describes the linear density field (again, the subscript "L" refers to "linear"):

$$\psi_L(y) \equiv \langle\mathrm{e}^{-y\delta_{L,R}}\rangle = \mathrm{e}^{y\langle\phi^2\rangle - \frac{1}{2}\mathrm{Tr}\ln(1+2i\Lambda_y\Delta_\phi)} \tag{60}$$

where the matrix Λ_y is given by:

$$\Lambda_y(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv -i \ y \ \frac{\theta(x_1 < R)}{V} \ \delta_D(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{61}$$

The normalization of the result derived from the path-integral in Eq. (60) is obtained from the condition $\psi_L(0) = 0$, as implied by the definition of $\psi_L(y)$ (since $\langle 1 \rangle = 1$). As shown in Eq. (4) the generating function $\psi_L(y)$ gives the moments $\langle\delta_{L,R}^q\rangle$ of the pdf $\mathcal{P}_L(\delta_{L,R})$ of the linear density contrast. However, it is usually convenient to introduce the generating function $\Phi_L(y)$ of the cumulants $\langle\delta_{L,R}^q\rangle_c$ which is given by (see any textbook on statistical theory):

$$\Phi_L(y) \equiv \ln[\psi_L(y)] = \sum_{q=1}^\infty \frac{(-y)^q}{q!} \langle\delta_{L,R}^q\rangle_c. \tag{62}$$

Thus, from Eq. (60) we obtain the exact expression:

$$\Phi_L(y) = \langle\phi^2\rangle y - \frac{1}{2} \ \mathrm{Tr} \ \ln(1 + 2i\Lambda_y\Delta_\phi). \tag{63}$$

One can easily check by expanding the logarithm over y that $\Phi_L(y)$ is actually quadratic over y (see also Eq. (75) below). Hence $\langle\delta_{L,R}\rangle = 0$ as it should. Moreover, we can check that we recover Eq. (49) for the variance of the linear density contrast.

4.3. Saddle-point contributions

In the non-linear case it is not so easy to compute the expression (58) since the integration over $\delta_L(\boldsymbol{x})$ no longer yields a Dirac functional. Besides, even for the linear density field the expression (63) is not very convenient since it is not obvious to compute the logarithm. Thus, for practical purposes it only provides the first few cumulants of $\delta_{L,R}$, which can be obtained by expanding the logarithm as a series over y and computing the trace of each power of $\Lambda_y\Delta_\phi$. Hence we can try a steepest-descent approximation to the path-integral (58), in the spirit of the calculations presented in Paper II for the Gaussian case or in Sect. 3 for a simple non-Gaussian model. More precisely, we may try to evaluate the contributions of all possible saddle-points to the path-integral (58).

First, we can look for the saddle-points of the exponent with respect to $\delta_{\mathrm{L}}(\boldsymbol{x})$. This yields:

$$i\lambda(\boldsymbol{x}) = y \, \frac{\delta(\delta_R)}{\delta(\delta_{\mathrm{L}}(\boldsymbol{x}))}. \tag{64}$$

Since the Gaussian random field $\phi(\boldsymbol{x})$ is homogeneous and isotropic we still have a spherical symmetry and we can again look for a spherical saddle-point. Then, taking the mean of Eq. (64) over a spherical cell V' we get:

$$i\lambda_{R'} = y \int_{V'} \frac{\mathrm{d}\boldsymbol{x}'}{V'} \, \frac{\delta(\delta_R)}{\delta(\delta_{\mathrm{L}}(\boldsymbol{x}'))}. \tag{65}$$

From the very definition of derivatives we can also write Eq. (65) as:

$$i\lambda_{R'} = \lim_{\epsilon \to 0} \frac{y}{\epsilon} \left\{ \delta_R \left[\delta_{\mathrm{L},R''} + \epsilon \Delta_\lambda(R', R'')\right] - \delta_R \left[\delta_{\mathrm{L},R''}\right] \right\} \tag{66}$$

where R'' is a dummy variable and we introduced the kernel:

$$\Delta_\lambda(R', R'') = \int_{V''} \frac{\mathrm{d}\boldsymbol{x}''}{V''} \, \frac{\theta(x'' < R')}{V'}$$
$$= \begin{cases} \frac{1}{V'} \text{ if } R'' < R' \\ \frac{1}{V''} \text{ if } R'' > R'. \end{cases} \tag{67}$$

We again define the linear density contrast $\delta_{R_{\mathrm{L}}}$ and the Lagrangian mass scale R_{L} as in Eq. (25) and we obtain:

$$i\lambda_{R'} = y \, \frac{\mathcal{F}'\left[\delta_{\mathrm{L},R_{\mathrm{L}}}\right] \Delta_\lambda(R', R_{\mathrm{L}})}{1 - \mathcal{F}'\left[\delta_{\mathrm{L},R_{\mathrm{L}}}\right] \frac{R^3}{3R_{\mathrm{L}}^2} \left.\frac{\mathrm{d}\delta_{\mathrm{L},R''}}{\mathrm{d}R''}\right|_{R_{\mathrm{L}}}} \tag{68}$$

in a fashion similar to the calculation presented in Paper II for the Gaussian case. The only term in the r.h.s. of Eq. (68) which depends on R' is the factor $\Delta_\lambda(R', R_{\mathrm{L}})$ hence we have:

$$\lambda_{R'} \propto \Delta_\lambda(R', R_{\mathrm{L}}) \tag{69}$$

which yields the profile:

$$\lambda(\boldsymbol{x}) = \lambda_0 \, \frac{\theta(x < R_{\mathrm{L}})}{V_{\mathrm{L}}} \tag{70}$$

which also defines the normalization λ_0. Note that this profile for the field $\lambda(\boldsymbol{x})$ is of the same form as the one obtained for the linear regime which involved the Dirac functional $\delta_{\mathrm{D}}[\lambda(\boldsymbol{x}) + iy\theta(x < R)/V]$, see the derivation of Eq. (60).

Then, for such a state $\lambda(\boldsymbol{x})$ we can estimate the trace which appeared in Eq. (58) as follows. Let us note this trace as $T(\lambda_0)$. Then, we have:

$$T(\lambda_0) \equiv \mathrm{Tr} \, \ln(1 + 2i\Lambda\Delta_\phi) = \sum_{q=1}^{\infty} \frac{(-1)^{q+1}}{q} \, (2i)^q \, T_q \tag{71}$$

with:

$$T_q \equiv \mathrm{Tr} \, (\Lambda\Delta_\phi)^q$$
$$= \int \mathrm{d}\boldsymbol{x}_1...\mathrm{d}\boldsymbol{x}_q \, (\Lambda\Delta_\phi)(\boldsymbol{x}_1, \boldsymbol{x}_2)...(\Lambda\Delta_\phi)(\boldsymbol{x}_q, \boldsymbol{x}_1) \tag{72}$$

using the definition of the trace: $\mathrm{Tr}M = \int \mathrm{d}\boldsymbol{x}M(\boldsymbol{x}, \boldsymbol{x})$. Next, from Eq. (56) we see that for the spherical state (70) the matrix $(\Lambda\Delta_\phi)$ is given by:

$$(\Lambda\Delta_\phi)(\boldsymbol{x}_1, \boldsymbol{x}_2) = \lambda_0 \, \frac{\theta(x_1 < R_{\mathrm{L}})}{V_{\mathrm{L}}} \, \Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{73}$$

which yields:

$$T_q = \lambda_0^q \int_{V_{\mathrm{L}}} \frac{\mathrm{d}\boldsymbol{x}_1}{V_{\mathrm{L}}}...\frac{\mathrm{d}\boldsymbol{x}_q}{V_{\mathrm{L}}} \, \Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2)...\Delta_\phi(\boldsymbol{x}_q, \boldsymbol{x}_1). \tag{74}$$

The first term T_1 (i.e. $q = 1$) is simply:

$$T_1 = \lambda_0 \int_{V_{\mathrm{L}}} \frac{\mathrm{d}\boldsymbol{x}_1}{V_{\mathrm{L}}} \, \Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_1) = \lambda_0 \langle\phi^2\rangle \tag{75}$$

where we used Eq. (46). Note that it is set by the short-distance behaviour ($x \ll x_{\mathrm{c}}$) of the two-point correlation Δ_ϕ, since we actually have $x = 0$. On the other hand, for $n_\phi < -3/2$ the other terms T_q with $q \geq 2$ are set by the long-distance behaviour of the correlation Δ_ϕ, i.e. by the scale R one is looking at by studying δ_R. This is the case for the power-spectra P_ϕ of cosmological interest, see Eq. (43). Moreover, for large q the different factors Δ_ϕ in Eq. (74) are almost uncorrelated and it makes sense to use the approximation:

$$q \geq 2: \ T_q \simeq \lambda_0^q \, \left[\sigma_\phi^2(R_{\mathrm{L}})\right]^q. \tag{76}$$

Besides, for a power-law power-spectrum $P_\phi(k)$ as in Eq. (43) we have (at scales larger than the coherence length x_{c}):

$$\Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2) \propto |\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-(n_\phi+3)}. \tag{77}$$

Then, since $n_\phi = -2.4$ is close to -3 the correlation $\Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is almost constant over most of the integration volume so that the approximation (76) should be quite reasonable. This yields for the trace $T(\lambda_0)$ defined in Eq. (71):

$$T(\lambda_0) \simeq 2i\lambda_0\langle\phi^2\rangle - 2i\lambda_0\sigma_\phi^2(R_{\mathrm{L}}) + \ln\left[1 + 2i\lambda_0\sigma_\phi^2(R_{\mathrm{L}})\right]. \tag{78}$$

Next, we could try to consider a saddle-point with respect to both $\delta_{\mathrm{L}}(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$. However, one can check that this procedure does not give meaningful results. In fact, as described above in the derivation of the exact Eq. (60) the generating function $\psi_{\mathrm{L}}(y)$ of the linear density field itself is not governed by a unique saddle-point. Indeed, we had to take into account the contributions from all real fields $\delta_{\mathrm{L}}(\boldsymbol{x})$ in the path-integral to get the Dirac functional which eventually provides the correct result. Then, in order to keep this structure while using the spherical saddle-points (70) it is natural to try to approximate the path-integral (58) by:

$$\psi(y) \simeq \int_{-\infty}^{\infty} \frac{\mathrm{d}\delta_{\mathrm{L},R_{\mathrm{L}}}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda_0 \, \mathrm{e}^{-\frac{1}{2} \mathrm{Tr} \, \ln(1+2i\Lambda\Delta_\phi)}$$
$$\times \, \mathrm{e}^{-y\delta_R(\delta_{\mathrm{L},R_{\mathrm{L}}}) + i\lambda_0(\langle\phi^2\rangle + \delta_{\mathrm{L},R_{\mathrm{L}}})}. \tag{79}$$

Here $\delta_R(\delta_{\mathrm{L},R_{\mathrm{L}}}) = \mathcal{F}(\delta_{\mathrm{L},R_{\mathrm{L}}})$ is given by the spherical collapse solution of the equation of motions, while the matrix

$\Lambda\Delta_\phi$ is given by Eq. (73). Thus, we have replaced the integration over the real fields $\delta_L(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ in Eq. (58) by an integration over the real numbers δ_{L,R_L} and λ_0. In other words, we have replaced the average over $\delta_L(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ by an average over spherical saddle-points of the form (70). In the case of the linear regime when $\delta(\boldsymbol{x}) = \delta_L(\boldsymbol{x})$ we simply have $\delta_R = \delta_{L,R_L}$ and $R_L = R$. Then we can perform the integration over δ_{L,R_L} which yields the Dirac function $\delta_D[\lambda_0 + iy]$. The integration over λ_0 is now straightforward and we recover exactly the rigorous result (60) (this also justifies the normalization factor 2π we introduced in Eq. (79)). Note in addition that the structure of the calculation is also the same (the Dirac functional is simply replaced by a Dirac function). Thus, the expression (79) is actually exact in the linear regime. We show in Appendix A how to rigorously derive the expression (79) from Eq. (58) in the linear regime. We also explain how it can be derived for the fully non-linear density field in the quasi-linear limit. Thus, Eq. (79) is actually exact to leading order in the limit $\sigma \to 0$. More precisely, for the non-linear density field it yields the exact exponential cutoff of the pdf $\mathcal{P}(\delta_R)$ which dominates the dependence on σ but the normalization is only correct up to a multiplicative factor of order unity which may depend on δ_R but not on σ. This factor arises from the Jacobians J_a and J_b in Eq. (A.15). We shall come back to this point below in Eq. (92).

As explained above, the factor $\mathrm{Tr}\ln(1 + 2i\Lambda\Delta_\phi)$ is not very convenient for practical calculations hence it is natural to use the approximation (78) which was derived for such spherical fields $\lambda(\boldsymbol{x})$. This yields:

$$\psi(y) \simeq \int \frac{\mathrm{d}\delta_L \mathrm{d}\lambda}{2\pi} \mathrm{e}^{-y\delta_R + i\lambda\delta_L + i\lambda\sigma_\phi^2(R_L) - \frac{1}{2}\ln(1 + 2i\lambda\sigma_\phi^2(R_L))} \tag{80}$$

where we removed the subscripts of the integration variables δ_L and λ. Note that in this approximation the term $\langle\phi^2\rangle$ has disappeared. In the linear regime, where $\delta_R = \delta_L$ and $R_L = R$, we can again perform the integrations over δ_L and λ which yields:

$$\Phi_L(y) \simeq y\,\sigma_\phi^2(R) - \frac{1}{2}\ln\left(1 + 2y\,\sigma_\phi^2(R)\right). \tag{81}$$

Then, expanding the logarithm around $y = 0$ we obtain the cumulants $\langle\delta_{L,R}^q\rangle_c$ from Eq. (62). This leads to $\langle\delta_{L,R}\rangle_c = 0$ and:

$$q \geq 2: \ \langle\delta_{L,R}^q\rangle_c \simeq 2^{q-1}\,(q-1)!\,\sigma_\phi^{2q}(R). \tag{82}$$

In particular, we obtain for the variance: $\sigma^2(R) \simeq 2\sigma_\phi^4(R)$, which must be compared with Eqs. (47) and (49). Of course, the deviation from the exact result (49) is entirely due to the approximation (76). For a power-law power-spectrum P_ϕ as in Eq. (43) with $n_\phi = -2.4$ we obtain for this deviation $\mathcal{D}$:

$$\mathcal{D} \equiv \frac{\sigma^2}{2\sigma_\phi^4} = \frac{\int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V}\frac{\mathrm{d}\boldsymbol{x}_2}{V}\,\Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2)^2}{\left[\int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V}\frac{\mathrm{d}\boldsymbol{x}_2}{V}\,\Delta_\phi(\boldsymbol{x}_1, \boldsymbol{x}_2)\right]^2} \simeq 1.11 \tag{83}$$

where we used Eq. (77) and the result (see Peebles & Groth 1976):

$$\int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V}\frac{\mathrm{d}\boldsymbol{x}_2}{V}\,|\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-\gamma} = \frac{72\,(2R)^{-\gamma}}{(3-\gamma)(4-\gamma)(6-\gamma)}. \tag{84}$$

Thus, we see that the approximation (76) is quite reasonable. Note that it becomes exact in the limit $n_\phi \to -3$. Then, from Eq. (82) we get for the skewness and the kurtosis:

$$D_3 \equiv \frac{\langle\delta_{L,R}^3\rangle_c}{\sigma^3} \simeq 2.83, \ D_4 \equiv \frac{\langle\delta_{L,R}^4\rangle_c}{\sigma^4} \simeq 12 \tag{85}$$

since we have:

$$\langle\delta_{L,R}^3\rangle_c \simeq 8\,\sigma_\phi^6, \ \langle\delta_{L,R}^4\rangle_c \simeq 48\,\sigma_\phi^8. \tag{86}$$

On the other hand, for the same power-spectrum ($n_\phi = -2.4$) Peebles (1999b) obtains by a direct numerical calculation (i.e. without the approximation (76)):

$$D_3 \simeq 2.46, \ D_4 \simeq 9.87 \tag{87}$$

and:

$$\frac{\langle\delta_{L,R}^3\rangle_c}{8\,\sigma_\phi^6} \simeq 1.02, \ \frac{\langle\delta_{L,R}^4\rangle_c}{48\,\sigma_\phi^8} \simeq 1.02. \tag{88}$$

Thus, we see that the approximation (76) is quite satisfactory. In particular, the behaviour of the high-order cumulants $\langle\delta_{L,R}^q\rangle_c$ is very well reproduced as soon as $q \gtrsim 3$. Then, we can expect the expression (80) to provide a good approximation to $\psi(y)$, both for the linear and the quasi-linear regimes, since we do not add any approximation in order to describe the non-linear effects encoded in the relations $\delta_{L,R} \leftrightarrow \delta_R$ and $R_L \leftrightarrow R$.

Going back to Eq. (80), we can make the change of variable $\lambda \to -iz/\sigma_\phi^2(R_L)$ which yields:

$$\psi(y) \simeq \int_{-\infty}^{\infty} \frac{\mathrm{d}\delta_L}{\sigma_\phi^2(R_L)} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}z}{2\pi i}\,\frac{\mathrm{e}^{-y\delta_R + z[1 + \delta_L/\sigma_\phi^2(R_L)]}}{\sqrt{1 + 2z}}. \tag{89}$$

Then, after a change of variable and using the integral representation of the Euler Gamma function:

$$\frac{1}{\Gamma(\nu)} = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}z}{2\pi i}\,z^{-\nu}\,\mathrm{e}^z \tag{90}$$

we obtain:

$$\psi(y) \simeq \int_{-\infty}^{\infty} \frac{\mathrm{d}\delta_L}{\sqrt{2\pi}\sigma_\phi^2(R_L)}\,\theta\left(1 + \frac{\delta_L}{\sigma_\phi^2(R_L)} > 0\right)$$
$$\times\,\mathrm{e}^{-y\delta_R}\,\mathrm{e}^{-\frac{1}{2}[1 + \delta_L/\sigma_\phi^2(R_L)]}\left[1 + \frac{\delta_L}{\sigma_\phi^2(R_L)}\right]^{-1/2} \tag{91}$$

where the factor θ is the Heaviside function with obvious notations. Let us recall that in Eq. (91) we have: $\delta_R(\delta_L) = \mathcal{F}(\delta_L)$ and $R_L = R(1 + \delta_R)^{1/3}$.

4.4. The pdf $\mathcal{P}_L(\delta_{L,R})$ and $\mathcal{P}(dR)$

In fact, since the generating function is written in an integral form we can directly derive the pdf $\mathcal{P}(\delta_R)$ using the inverse Laplace transform (3). Indeed, the integration over y simply gives the Dirac function $\delta_D[\delta_R - \mathcal{F}(\delta_L)]$. After a trivial integration over δ we obtain:

$$
\mathcal{P}(\delta_R) \simeq \theta\left(1 + \frac{\delta_L}{\sigma_\phi^2(R_L)} > 0\right) \frac{1}{\sqrt{2\pi}\sigma_\phi^2(R_L)} \frac{1}{\mathcal{F}'(\delta_L)}
$$
$$
\times \left[1 + \frac{\delta_L}{\sigma_\phi^2(R_L)}\right]^{-1/2} e^{-\frac{1}{2}[1+\delta_L/\sigma_\phi^2(R_L)]} \quad (92)
$$

where the quantity δ_L is given by the condition: $\mathcal{F}(\delta_L) = \delta_R$ and $R_L = R(1+\delta_R)^{1/3}$. In particular, the pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly evolved density field is given by:

$$
\mathcal{P}_L(\delta_{L,R}) \simeq \theta\left(1 + \frac{\delta_{L,R}}{\sigma_\phi^2(R)} > 0\right) \frac{1}{\sqrt{2\pi}\sigma_\phi^2(R)}
$$
$$
\times \left[1 + \frac{\delta_{L,R}}{\sigma_\phi^2(R)}\right]^{-1/2} e^{-\frac{1}{2}[1+\delta_{L,R}/\sigma_\phi^2(R)]}. \quad (93)
$$

It is obtained from Eq. (92) by setting $\mathcal{F}(\delta_L) = \delta_L$ and $R_L = R$. For Gaussian initial conditions the pdf $\mathcal{P}_L(\delta_{L,R})$ exhibits a specific scaling over the variable $\nu = \delta_{L,R}/\sigma(R)$, which contains all the time and scale dependence of the pdf. As described above, within the approximation (76) we have $\sigma^2 \simeq 2\sigma_\phi^4$ hence we get:

$$
\nu \equiv \frac{\delta_L}{\sigma(R)} = \frac{\delta_L}{\sqrt{2}\sigma_\phi^2(R)}. \quad (94)
$$

Then, we see that the linear pdf (93) shows a scaling over ν as in the Gaussian case:

$$
\mathcal{P}_L(\delta_{L,R}) \, d\delta_{L,R} = \mathcal{P}_L^{(\nu)}(\nu) \, d\nu, \quad (95)
$$

but the pdf $\mathcal{P}_L^{(\nu)}$ is now given by:

$$
\mathcal{P}_L^{(\nu)}(\nu) \equiv \frac{\theta(1+\sqrt{2}\nu > 0)}{\sqrt{\pi}} \frac{1}{\sqrt{1+\sqrt{2}\nu}} e^{-\frac{1}{2}(1+\sqrt{2}\nu)}. \quad (96)
$$

As noticed in Paper II, in the Gaussian case the non-linear high-density tail is well described by a simple spherical model. In fact, this model gives the exact exponential dependence of the pdf on the variance σ, at leading order in the limit $\sigma \to 0$, as was also shown in Valageas (1998). Let us recall here how to build this simple model. It is based on the approximation:

$$
\int_{\delta_R}^\infty d\delta \, (1+\delta)\mathcal{P}(\delta) \simeq \int_{\delta_{L,R_L}}^\infty d\delta_L \, \mathcal{P}_L(\delta_L) \quad (97)
$$

which merely states that the fraction of matter within spherical cells of radius R with a non-linear density contrast larger than δ_R is approximately equal to the fraction of matter which was originally enclosed within spherical cells of radius R_L with a linear density contrast larger than

δ_{L,R_L}. Here R_L and δ_{L,R_L} are related to the non-linear variables R and δ_R by the usual relation (25), as in Eq. (92). Note that this is similar to the usual Press-Schechter prescription (Press & Schechter 1974), without the factor 2. Then, substituting the scaling (95) into Eq. (97) and differentiating with respect to δ_R we obtain:

$$
\mathcal{P}_s(\delta_R) = \frac{1}{1+\delta_R} \frac{d\nu}{d\delta_R} \mathcal{P}_L^{(\nu)}(\nu), \quad \text{with} \quad \nu = \frac{\delta_{L,R_L}}{\sigma(R_L)}. \quad (98)
$$

Here the subscript "s" refers to the "spherical" model. In order to recover the variables used in the Gaussian case (Paper II) it is convenient to introduce the variable τ given by:

$$
\tau \equiv -\frac{\delta_{L,R_L}\sigma(R)}{\sigma(R_L)} = -\nu\,\sigma(R), \quad (99)
$$

which removes the dependence on the amplitude of the rms linear density fluctuation, and to define the function $\mathcal{G}(\tau)$ by:

$$
\mathcal{G}(\tau) \equiv \mathcal{F}[\delta_{L,R_L}] = \delta_R \quad (100)
$$

which obeys again Eq. (31). Then, the relation (98) writes:

$$
\mathcal{P}_s(\delta_R) = \frac{1}{1+\delta_R} \frac{1}{\sigma(R)} \frac{1}{|\mathcal{G}'(\tau)|} \mathcal{P}_L^{(\nu)}(\nu) \quad (101)
$$

which yields:

$$
\mathcal{P}_s(\delta_R) = \theta\left(1 + \frac{\delta_L}{\sigma_\phi^2(R_L)} > 0\right) \frac{1}{\sqrt{2\pi}\sigma_\phi^2(R_L)} \frac{1}{1+\delta_R}
$$
$$
\times \frac{1}{|\mathcal{G}'(\tau)|} \left[1 + \frac{\delta_L}{\sigma_\phi^2(R_L)}\right]^{-1/2} e^{-\frac{1}{2}[1+\delta_L/\sigma_\phi^2(R_L)]}. \quad (102)
$$

We can see that we recover the expression (92), except for the multiplicative factor where $\mathcal{F}'(\delta_{L,R_L})$ has been replaced by $(1+\delta_R)|\mathcal{G}'(\tau)|$. Thus, as in the Gaussian case, we find that the simple spherical model (97) yields the exact exponential σ-dependent term of the pdf. However, the multiplicative factor may not be exact. Here, we note that the multiplicative factor which appears in Eq. (92) is not exact either, as stated above, below Eq. (79), see also Appendix A. In the Gaussian case, we checked by a comparison with numerical simulations (e.g., Valageas 1998; Paper II) that the spherical model (97) provides good results up to $\sigma \simeq 1$. We can expect a similar accuracy in the non-Gaussian case studied here. Therefore, the expression (102) should give a good description of the non-linear pdf. This means that the multiplicative factor which appears in Eq. (102) gives a reasonable approximation of the term induced by the Jacobians which arise in the non-linear case, see Eq. (A.15).

We show in Fig. 2 our results for $n_\phi = -2.4$ and $\sigma = 0.74$. The dashed-line shows the pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly evolved density field, from Eq. (93). As could be seen from Eq. (93) it is strongly non-Gaussian, whatever the value of σ_ϕ. In particular, for large overdensities $\delta_{L,R}$ the pdf exhibits a simple exponential cutoff (multiplied by a power $\delta_{L,R}^{-1/2}$) which is much smoother than
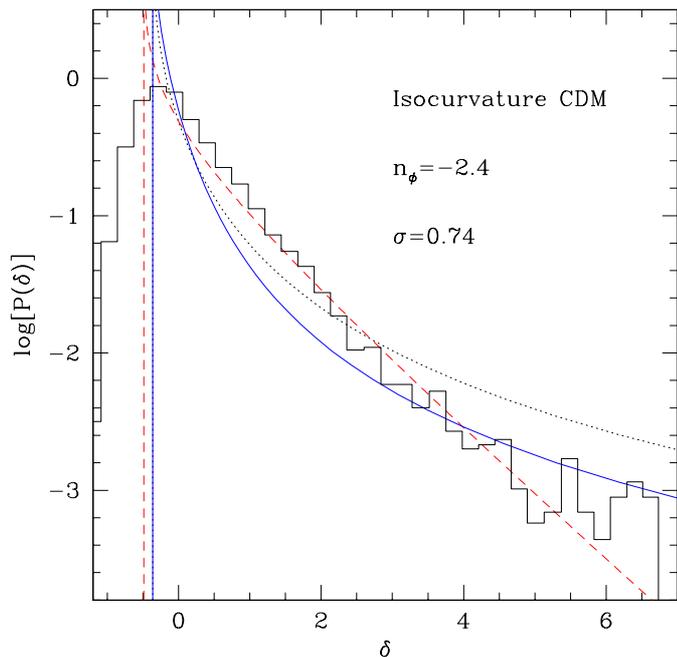
Fig. 2. The pdf $\mathcal{P}(\delta_R)$ for the isocurvature CDM scenario, with $n_\phi = -2.4$ and $\sigma = 0.74$. This corresponds to $\sigma_\phi = 0.70$. The solid line shows the theoretical prediction from (102) for $\mathcal{P}(\delta_R)$. This corresponds to the "spherical" model. The dotted line is given by Eq. (102) where no attempt is made to take into account the Jacobians which arise in the non-linear case. The solid line (spherical model) should give better results. The dashed-line displays the pdf $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L},R})$ of the linearly evolved density field, from Eq. (93). The histogram shows the results of numerical simulations by Robinson & Baker (2000) for the linear pdf $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L},R})$.

a Gaussian cutoff. Hence the number of extreme events (i.e. $\delta_{\mathrm{L},R} \gg \sigma$) is much larger than for the Gaussian case, as can be checked by comparison with the lower dotted curve in Fig. 1. We can see that for $\delta_{\mathrm{L},R} \geq 0$ our predictions agree very well with the results of numerical simulations from Robinson & Baker (2000) for the linear density field (shown by the histogram). Note that for the theoretical curve we used the exact value of σ_ϕ obtained from Eq. (83) in order to get a meaningful comparison. Hence the variance of the approximate pdf (93) underestimates the exact result by a factor $\mathcal{D} \simeq 1.11$. However, we can check that the agreement is quite good for $\delta_{\mathrm{L},R} \geq 0$ so that for practical purposes one can directly use Eq. (93). For instance, Robinson & Baker (2000) actually use the linear pdf to apply the usual Press-Schechter recipe (Press & Schechter 1974) in order to estimate the mass function of just-collapsed objects.

On the other hand, we note that our estimate (93) fails for $\delta_{\mathrm{L},R} \lesssim 0$. In particular, we see that the pdf vanishes for $\delta_{\mathrm{L},R} < -\sigma_\phi^2(R)$, that is the exact lower bound $\delta_{\mathrm{L},R} \geq -\langle\phi^2\rangle$ has been replaced by $\delta_{\mathrm{L},R} \geq -\sigma_\phi^2(R)$. This is due to the approximation (76). Indeed, the exact linear pdf

obtained from Eqs. (3) and (60) is:

$$\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L},R}) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}y}{2\pi i}$$
$$\times \mathrm{e}^{y(\delta_{\mathrm{L},R}+\langle\phi^2\rangle)-\frac{1}{2}\mathrm{Tr}\,\ln(1+2i\Lambda_y\Delta_\phi)}. \tag{103}$$

This expression clearly shows that the linear density contrast obeys the lower bound $\delta_{\mathrm{L},R} \geq -\langle\phi^2\rangle$. Indeed, for $\delta_{\mathrm{L},R} < -\langle\phi^2\rangle$ we can push the integration path over y to the right (i.e. $\mathrm{Re}(y) \to +\infty$) so that the integral vanishes (we have $\Lambda_y \propto -iy$). Note that this behaviour is rather different from the Gaussian case where the linear density contrast can take negative values of arbitrarily large amplitude. In fact, the lower bound for $\delta_{\mathrm{L},R}$ can be directly seen from Eq. (45). Note that this bound is directly related to the time t_i at the end of inflation where Eq. (41) holds. Indeed, from Eq. (44) we see that $\langle\phi^2\rangle = D_+(t)$ where the growing mode has been normalized by $D_+(t_i) = 1$. This minimum value of $\delta_{\mathrm{L},R}$ is also overestimated by the numerical simulations which only start at a redshift $z_i \sim 39$. Nevertheless, it is clear that the approximate bound $\delta_{\mathrm{L},R} \geq -\sigma_\phi^2$ gives the value below which underdensities become very rare (i.e. the pdf should decline for $\delta_{\mathrm{L},R} \lesssim -\sigma_\phi^2$). Besides, for practical purposes one is often mainly interested in the behaviour of overdensities $\delta_R > 0$ so that the expressions (92) and (93) should remain useful.

As noticed in Eq. (95), the linear pdf given by Eq. (93) exhibits a scaling over the one variable ν defined in Eq. (94), as in the usual Gaussian case. This scaling property was actually used in order to obtain $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L},R})$ from the numerical simulations. However, this property is not exact and it fails for underdensities since as noticed above the linear density contrast also satisfies the lower bound $-\langle\phi^2\rangle$ which does not scale as σ (e.g., contrary to σ it is independent of scale).

Next, we show the theoretical prediction $\mathcal{P}(\delta_R)$ from Eq. (92) (dotted line) and from Eq. (102) (solid line) from the spherical model (97) for the actual non-linear density field. In Fig. 2 we used the approximation:

$$\mathcal{F}(\delta) \simeq \left(1 - \frac{\delta}{1.5}\right)^{-1.5} - 1 \tag{104}$$

which has been shown to provide a very good fit to the exact spherical collapse solution for all values of Ω_{m} and Ω_Λ, see Fig. 2 in Bernardeau (1994b). This means that the dependence on Ω_{m} and Ω_Λ is negligible (in the quasi-linear regime where we do not consider virialized objects) and the pdf only depends on the rms fluctuation σ (and the slope n_ϕ of the power-spectrum). Unfortunately, there are no available results from numerical simulations to compare with our prediction. However, in view of the reasonable accuracy of our prediction for the linear pdf $\mathcal{P}_{\mathrm{L}}(\delta_{\mathrm{L},R})$ and the good results obtained for the Gaussian case (see Paper II) we can expect the expression (102) to provide a good estimate of the exact pdf $\mathcal{P}(\delta_R)$. It should perform better than Eq. (92) where no attempt was made to take into account the Jacobians which arise in the non-linear

case, see Appendix A. Of course, we recover the usual features of the non-linear evolution which increases the high-density tail of the pdf. Note again the much larger number of high density events with respect to the Gaussian case with the same variance (see Fig. 1).

5. Standard slow-roll inflation

5.1. Initial conditions

The two classes of models we investigated in Sects. 3 and 4 are strongly non-Gaussian (except for $\alpha \simeq 2$ in the first case) and they apply to "non-standard" scenarios for the generation of primordial density fluctuations. For instance, the second model is based on a multifield inflationary scenario. By contrast, in standard slow-roll inflation the primordial density perturbations should be very close to Gaussian. Nevertheless, they may still show some small deviations from Gaussianity. In particular, one is led to consider models where the perturbed primordial gravitational potential $\Phi(\boldsymbol{x})$ is given by:

$$\Phi(\boldsymbol{x}) = \phi(\boldsymbol{x}) + \epsilon\left(\phi(\boldsymbol{x})^2 - \langle\phi^2\rangle\right) \tag{105}$$

where $\phi(\boldsymbol{x})$ is a Gaussian random field with zero mean and power-spectrum $P_\phi(k)$, as in Eq. (42). Such models arise in standard slow-roll inflation if we keep track of the perturbations of the inflaton up to the second-order (e.g., Gangui et al. 1994; Falk et al. 1993). In fact, Eq. (105) can be seen as the first two terms of a Taylor expansion so that it applies to a large class of slightly non-Gaussian models. Thus, the parameter ϵ is taken to be small so that the primordial density fluctuations are close to Gaussian. Note that ϵ has the dimensions of ϕ^{-1}. We shall define below what we mean by "ϵ being small" (see Eq. (138)). Then, our goal here is to obtain the pdf $\mathcal{P}(\delta_R)$ of the density fluctuations in the quasi-linear regime up to first order in ϵ (to go beyond this order we should first take into account the possible higher-order terms in Eq. (105)). Note that for $\epsilon = 0$ we must recover the case of Gaussian primordial density fluctuations studied in Paper II. The gravitational potential $\Phi(\boldsymbol{x})$ is related to the primordial linear density fluctuations $\delta_L(\boldsymbol{x})$ by the Poisson equation (in comoving coordinates):

$$\Delta\Phi = \frac{4\pi\mathcal{G}\overline{\rho}}{a}\,\delta_L(\boldsymbol{x}) \tag{106}$$

where $a(t)$ is the scale-factor and $\overline{\rho}$ the mean comoving density. Taking the Fourier transform of Eq. (106) one can eventually write the linear density field $\delta_L(\boldsymbol{x})$ at the time of interest as:

$$\delta_L(\boldsymbol{x}) = \int d\boldsymbol{x}'\,W(\boldsymbol{x},\boldsymbol{x}')\left[\phi(\boldsymbol{x}') + \epsilon\left(\phi(\boldsymbol{x}')^2 - \langle\phi^2\rangle\right)\right] \tag{107}$$

where the kernel $W(\boldsymbol{x},\boldsymbol{x}')$ is given by:

$$W(\boldsymbol{x},\boldsymbol{x}') = -\frac{2}{3}\Omega_m^{-1}H_0^{-2}D_+(t)\int\frac{d\boldsymbol{k}}{(2\pi)^3}$$
$$\times e^{i\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')}k^2 T(k). \tag{108}$$

Here we introduced the Hubble constant H_0 and the density parameter Ω_m today, while $D_+(t)$ is the usual linear growing mode at the time of interest, normalized by $D_+(t_0) = 1$ today. The function $T(k)$ is simply the adiabatic CDM transfer function (normalized to unity for $k \to 0$). Indeed, contrary to the isocurvature model described in Sect. 4 we need to take into account the deviations of $T(k)$ from unity on the scales of interest for large-scale structure formation. The factor k^2 arises from the Laplacian in the l.h.s. in Eq. (106). Note that the kernel $W(\boldsymbol{x},\boldsymbol{x}')$ is homogeneous, isotropic and symmetric since it only depends on $|\boldsymbol{x} - \boldsymbol{x}'|$. Hence for any real fields f_1 and f_2 we have:

$$(f_1.W.f_2) = (f_2.W.f_1) = \int d\boldsymbol{x}_1 d\boldsymbol{x}_2 W(\boldsymbol{x}_1,\boldsymbol{x}_2)$$
$$\times f_1(x_1)f_2(x_2). \tag{109}$$

Thus, Eq. (107) defines the initial conditions of our system.

We can note that the pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly evolved density field was already studied in Matarrese et al. (2000) using path-integral methods like those described in Sect. 5.2 below. However, they did not investigate the effects of the non-linear dynamics. On the other hand, Verde et al. (2000) studied the observational tests which may constrain such deviations from Gaussian initial conditions.

5.2. Generating function

In order to derive the pdf $\mathcal{P}(\delta_R)$ we again introduce the Laplace transform $\psi(y)$ as in Eq. (2), which yields again Eq. (52). Next, we introduce the linear density field through the Dirac functional $\delta_D[\delta_L(\boldsymbol{x}) - \int d\boldsymbol{x}' W(\boldsymbol{x},\boldsymbol{x}')\{\phi(\boldsymbol{x}') + \epsilon(\phi(\boldsymbol{x}')^2 - \langle\phi^2\rangle)\}]$. This Dirac functional can again be expressed through an auxiliary real field $\lambda(\boldsymbol{x})$ so that the analog of Eq. (54) now reads:

$$\psi(y) = \int[d\phi(\boldsymbol{x})][d\delta_L(\boldsymbol{x})][d\lambda(\boldsymbol{x})]\,e^{-y\delta_R[\delta_L]-\frac{1}{2}\phi.\Delta_\phi^{-1}.\phi}$$
$$\times e^{i\lambda.(\delta_L - W.[\phi + \epsilon(\phi^2 - \langle\phi^2\rangle)])} \tag{110}$$

where we defined the inverse Δ_ϕ^{-1} of the two-point correlation Δ_ϕ of the Gaussian random field ϕ, which is again given by Eq. (46). The mean $\langle\phi^2\rangle$ is again given by Eq. (48) and it does not depend on the field $\phi(\boldsymbol{x})$ over which we integrate. As in Sect. 4 we do not write explicitly the normalization factor of the path-integrals. Next, we introduce the kernel $\Lambda_W(\boldsymbol{x}_1,\boldsymbol{x}_2)$ defined by:

$$\Lambda_W(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \delta_D(\boldsymbol{x}_1 - \boldsymbol{x}_2)\int d\boldsymbol{x}\,\lambda(\boldsymbol{x})W(\boldsymbol{x},\boldsymbol{x}_1) \tag{111}$$

so that Eq. (110) reads:

$$\psi(y) = \int[d\phi(\boldsymbol{x})][d\delta_L(\boldsymbol{x})][d\lambda(\boldsymbol{x})]\,e^{-y\delta_R[\delta_L]-\frac{1}{2}\phi.\Delta_\phi^{-1}.\phi}$$
$$\times e^{i\lambda.\delta_L - i\lambda.W.\phi - i\epsilon\phi.\Lambda_W.\phi + i\epsilon\langle\phi^2\rangle(\lambda.W.1)}. \tag{112}$$

As explained in Sect. 5.1 we only consider the first-order term in ϵ. Therefore, we expand the exponential in Eq. (112) up to first-order in ϵ. This yields:

$$\psi(y) = \int [\mathrm{d}\phi(\boldsymbol{x})][\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})] \, \mathrm{e}^{-y\delta_R - i\lambda.W.\phi - \frac{1}{2}\phi.\Delta_\phi^{-1}.\phi}$$
$$\times \, \mathrm{e}^{i\lambda.\delta_{\mathrm{L}}} \left[1 + i\epsilon\langle\phi^2\rangle(\lambda.W.1) - i\epsilon(\phi.\Lambda_{\mathrm{W}}.\phi) \right]. \quad (113)$$

This path-integral is Gaussian over the random field ϕ. Thus, we first make the change of variable $\phi = \phi' - i\Delta_\phi.W.\lambda$ so that Eq. (113) writes:

$$\psi(y) = \int [\mathrm{d}\phi'(\boldsymbol{x})][\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})] \, \mathrm{e}^{-y\delta_R + i\lambda.\delta_{\mathrm{L}} - \frac{1}{2}\lambda.W.\Delta_\phi.W.\lambda}$$
$$\times \, \mathrm{e}^{-\frac{1}{2}\phi'.\Delta_\phi^{-1}.\phi'} \left[1 + i\epsilon\langle\phi^2\rangle(\lambda.W.1) - i\epsilon(\phi'.\Lambda_{\mathrm{W}}.\phi') \right.$$
$$\left. -2\epsilon(\phi'.\Lambda_{\mathrm{W}}.\Delta_\phi.W.\lambda) + i\epsilon(\lambda.W.\Delta_\phi.\Lambda_{\mathrm{W}}.\Delta_\phi.W.\lambda) \right]. \quad (114)$$

Here we used the fact that all kernels W, Δ_ϕ and Λ_{W} are symmetric. Then, using Wick's theorem we can perform the Gaussian integration over $\phi'(\boldsymbol{x})$ which yields:

$$\psi(y) = \int [\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})][\mathrm{d}\lambda(\boldsymbol{x})] \, \mathrm{e}^{-y\delta_R + i\lambda.\delta_{\mathrm{L}} - \frac{1}{2}\lambda.W.\Delta_\phi.W.\lambda}$$
$$\times \left[1 + i\epsilon(\lambda.W.\Delta_\phi.\Lambda_{\mathrm{W}}.\Delta_\phi.W.\lambda) \right] \quad (115)$$

where we used $\Delta_\phi(\boldsymbol{x},\boldsymbol{x}) = \langle\phi^2\rangle$. Next, we define the two-point correlation $\Delta_{\mathrm{L}0}(\boldsymbol{x}_1,\boldsymbol{x}_2)$ of the linear density field in the case $\epsilon = 0$ by:

$$\Delta_{\mathrm{L}0}(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \langle\delta_{\mathrm{L}}(\boldsymbol{x}_1)\delta_{\mathrm{L}}(\boldsymbol{x}_2)\rangle_0$$
$$= \langle(W.\phi)(\boldsymbol{x}_1)(W.\phi)(\boldsymbol{x}_2)\rangle \quad (116)$$

where we used Eq. (107). Here the subscript "0" refers to "$\epsilon = 0$". The relation (116) also writes:

$$\Delta_{\mathrm{L}0} = W.\Delta_\phi.W. \quad (117)$$

The path-integral (115) is Gaussian over the field $\lambda(\boldsymbol{x})$. Thus, making the change of variable $\lambda = \lambda' + i\Delta_{\mathrm{L}0}^{-1}.\delta_{\mathrm{L}}$ and integrating over $\lambda'(\boldsymbol{x})$ using Wick's theorem we finally get:

$$\psi(y) = \int [\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})] \, \mathrm{e}^{-y\delta_R[\delta_{\mathrm{L}}] - \frac{1}{2}\delta_{\mathrm{L}}.\Delta_{\mathrm{L}0}^{-1}.\delta_{\mathrm{L}}}$$
$$\times \left[1 + \epsilon \, (\Delta_\phi^{-1}.W^{-1}.\delta_{\mathrm{L}}).(W^{-1}.\delta_{\mathrm{L}})^2 \right] \quad (118)$$

where we used $\int \mathrm{d}\boldsymbol{x} \, \delta_{\mathrm{L}}(\boldsymbol{x}) = 0$. Here we introduced the short-hand notation $(W^{-1}.\delta_{\mathrm{L}})^2$ for the vector:

$$(W^{-1}.\delta_{\mathrm{L}})^2(\boldsymbol{x}) \equiv \left(\int \mathrm{d}\boldsymbol{x}' \, W^{-1}(\boldsymbol{x},\boldsymbol{x}')\delta_{\mathrm{L}}(\boldsymbol{x}') \right)^2. \quad (119)$$

Note that for $\epsilon = 0$ we recover the expression (14) derived for Gaussian initial conditions. This was to be expected since for $\epsilon = 0$ the linear density field is actually Gaussian. Indeed, in this case we have $\delta_{\mathrm{L}} = W.\phi$ and ϕ is a Gaussian random field. Of course, the procedure we described above can be extended up to any order in ϵ (provided we know the initial conditions up to the required order). Indeed, by expanding the exponential which appears in Eq. (110) up to the needed order in ϵ the integrations over the fields $\phi(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ are Gaussian and can be easily performed using Wick's theorem. Then, one eventually obtains an expression of the form (118), where the term in brackets is an expansion over ϵ up to the required order. Note that this is similar to the standard Edgeworth expansion where one directly expands the Laplace transform $\psi(y)$ around the Gaussian value $\psi_0(y) = \mathrm{e}^{y^2\sigma^2/2}$ (see any textbook on probability theory). Therefore, the expansion over ϵ does not necessarily converge: it may only provide an asymptotic series (e.g., Cramer 1946).

5.3. Steepest-descent method

We now need to evaluate the path-integral (118). As for the Gaussian case studied in Paper II or the non-Gaussian generalization described in Sect. 3 we can use a steepest-descent method for the quasi-linear regime. We refer the reader to Paper II for a detailed presentation of this method. First, we define the rescaled generating function $\overline{\psi}(y)$ by:

$$\overline{\psi}(y) \equiv \psi\left(y/\sigma_0(R)^2\right) \quad (120)$$

where $\sigma_0(R)$ is the variance of the linear density field for $\epsilon = 0$:

$$\sigma_0(R)^2 \equiv \langle\delta_{\mathrm{L},R}^2\rangle_0 = \int_V \frac{\mathrm{d}\boldsymbol{x}_1}{V} \frac{\mathrm{d}\boldsymbol{x}_2}{V} \, \Delta_{\mathrm{L}0}(\boldsymbol{x}_1,\boldsymbol{x}_2). \quad (121)$$

Using Eq. (118) we get:

$$\overline{\psi}(y) = \int [\mathrm{d}\delta_{\mathrm{L}}(\boldsymbol{x})] \, \mathrm{e}^{-S[\delta_{\mathrm{L}}]/\sigma_0(R)^2}$$
$$\times \left[1 + \epsilon \, (\Delta_\phi^{-1}.W^{-1}.\delta_{\mathrm{L}}).(W^{-1}.\delta_{\mathrm{L}})^2 \right] \quad (122)$$

where we introduced the "action" $S[\delta_{\mathrm{L}}]$ given by:

$$S[\delta_{\mathrm{L}}] \equiv y \, \delta_R[\delta_{\mathrm{L}}] + \frac{\sigma_0(R)^2}{2} \, \delta_{\mathrm{L}}.\Delta_{\mathrm{L}0}^{-1}.\delta_{\mathrm{L}}. \quad (123)$$

In the quasi-linear limit $\sigma_0 \to 0$ the path-integral (122) is governed by the minimum of the action S. As shown in Paper II this spherically symmetric saddle-point is given by:

$$\delta_{\mathrm{L}}(\boldsymbol{x}) = \delta_{\mathrm{L},R_{\mathrm{L}}} \int_{V_{\mathrm{L}}} \frac{\mathrm{d}\boldsymbol{x}'}{V_{\mathrm{L}}} \frac{\Delta_{\mathrm{L}0}(\boldsymbol{x},\boldsymbol{x}')}{\sigma_0(R_{\mathrm{L}})^2}$$
$$= \frac{\delta_{\mathrm{L},R_{\mathrm{L}}}}{\sigma_0(R_{\mathrm{L}})^2} \, (W.\Delta_\phi.W).\left(\frac{\theta(x' < R_{\mathrm{L}})}{V_{\mathrm{L}}} \right) \quad (124)$$

where $\boldsymbol{x}'$ is a dummy variable. The variable $\delta_{\mathrm{L},R_{\mathrm{L}}}$ is given by the implicit equation:

$$\delta_{\mathrm{L},R_{\mathrm{L}}} = -y \, \frac{\mathcal{F}'[\delta_{\mathrm{L},R_{\mathrm{L}}}]\sigma_0^2(R_{\mathrm{L}})/\sigma_0^2(R)}{1 - \mathcal{F}'[\delta_{\mathrm{L},R_{\mathrm{L}}}]\frac{R^3}{3R_{\mathrm{L}}^2}\delta_{\mathrm{L},R_{\mathrm{L}}}\frac{1}{\sigma_0(R_{\mathrm{L}})}\frac{\mathrm{d}\sigma_0}{\mathrm{d}R}(R_{\mathrm{L}})}. \quad (125)$$

Here the function $\mathcal{F}(\delta_{\mathrm{L},R_{\mathrm{L}}})$ is the usual spherical collapse solution while R_{L} is the Lagrangian mass scale, see Eq. (25). These results were also used in Sect. 3.2

where we investigated a simple generalization of the Gaussian weight. Next, we introduce the functions $\tau(\delta_{\mathrm{L}})$ and $\mathcal{G}(\tau)$ by:

$$\tau(\delta_{\mathrm{L}}) \equiv \frac{-\delta_{\mathrm{L}}\,\sigma_0(R)}{\sigma_0\left[(1+\mathcal{F}[\delta_{\mathrm{L}}])^{1/3}R\right]} \tag{126}$$

and:

$$\mathcal{G}(\tau) \equiv \mathcal{F}\left[\delta_{\mathrm{L}}(\tau)\right] = \delta_R \tag{127}$$

as in Eqs. (29) and (30). Then, we define the function $\varphi_0(y) \equiv \min S[\delta_{\mathrm{L}}]$ as the value of the action S at this spherically symmetric saddle-point (i.e. the minimum of the action). As shown in Paper II it is given by the implicit system:

$$\begin{cases} \tau = -y\,\mathcal{G}'(\tau) \\[2mm] \varphi_0(y) = y\,\mathcal{G}(\tau) + \dfrac{\tau^2}{2} \end{cases} \tag{128}$$

in a fashion similar to Eq. (32). Then, at leading order in σ_0 we can write the path-integral (122) in the quasi-linear limit as:

$$\overline{\psi}(y) = \mathrm{e}^{-\varphi_0(y)/\sigma_0(R)^2}\left[1 - \epsilon\,\frac{\tau^3}{\sigma_0(R)^3}\,\frac{J_3(R_{\mathrm{L}})}{\sigma_0(R_{\mathrm{L}})^3}\right] \tag{129}$$

where we defined:

$$J_3(R_{\mathrm{L}}) \equiv \left(W.\frac{\theta(x_1<R_{\mathrm{L}})}{V_{\mathrm{L}}}\right).\left(\Delta_\phi.W.\frac{\theta(x_2<R_{\mathrm{L}})}{V_{\mathrm{L}}}\right)^2. \tag{130}$$

Here $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are dummy variables. The normalization of $\overline{\psi}(y)$ in Eq. (129) is set by the constraint $\overline{\psi}(0) = \langle 1\rangle = 1$. Next, we define the dimensionless quantity $\tilde{\epsilon}$ by:

$$\tilde{\epsilon}(R_{\mathrm{L}}) \equiv \epsilon\,\frac{J_3(R_{\mathrm{L}})}{\sigma_0(R_{\mathrm{L}})^3}. \tag{131}$$

It is convenient to express the quantities σ_0 and J_3 in terms of the power-spectrum $P_\phi(k)$ of the Gaussian random field $\phi(\boldsymbol{x})$. This yields:

$$\sigma_0(R_{\mathrm{L}})^2 \equiv (2\pi)^6 \int \mathrm{d}\boldsymbol{k}\,P_\phi(k)W(k)^2 F(kR_{\mathrm{L}})^2 \tag{132}$$

and:

$$J_3(R_{\mathrm{L}}) \equiv (2\pi)^9 \int \mathrm{d}\boldsymbol{k}\mathrm{d}\boldsymbol{k}'\,P_\phi(k)P_\phi(k')W(k)W(k')$$
$$\times\,W(|\boldsymbol{k}+\boldsymbol{k}'|)F(kR_{\mathrm{L}})F(k'R_{\mathrm{L}})F(|\boldsymbol{k}+\boldsymbol{k}'|R_{\mathrm{L}}). \tag{133}$$

Here $W(\boldsymbol{k}) = W(|\boldsymbol{k}|)$ is the Fourier transform of the window function $W(\boldsymbol{x})$ defined in Eq. (108) (i.e. it is related to the kernel W by $W(\boldsymbol{x}_1,\boldsymbol{x}_2) = W(\boldsymbol{x}_2-\boldsymbol{x}_1)$) while $F(kR)$ is the Fourier transform of the top-hat of radius R:

$$F(kR) \equiv \int_V \frac{\mathrm{d}\boldsymbol{x}}{V}\,\mathrm{e}^{i\boldsymbol{k}.\boldsymbol{x}} = 3\,\frac{\sin(kR)-(kR)\cos(kR)}{(kR)^3}. \tag{134}$$

Note that in Eq. (134) we did not introduce the factor $(2\pi)^{-3}$ used in the definition (6) of the Fourier transform, in order to obtain the usual top-hat window $F(kR)$. As in the previous sections we approximate the power-spectrum $P_0(k)$ of the linear density fluctuations (when $\epsilon = 0$) by a power-law. Since we have $P_\phi(k) \propto k^{-3}$ (i.e. a scale-invariant primordial power-spectrum, with small and large scale cutoffs) for the standard inflationary scenarios we investigate here this means that $W(k)^2 \propto k^{n+3}$ on the scales of interest. Indeed, the power-spectrum $P_0(k)$ is related to $P_\phi(k)$ by:

$$P_0(k) = (2\pi)^6 W(k)^2 P_\phi(k), \tag{135}$$

as can be seen for instance from Eq. (132). Then, from Eqs. (132) and (133) we get:

$$\sigma_0(R_{\mathrm{L}})^2 \propto R_{\mathrm{L}}^{-(n+3)}, \quad J_3(R_{\mathrm{L}}) \propto R_{\mathrm{L}}^{-3(n+3)/2} \tag{136}$$

which implies that the quantity $\tilde{\epsilon}(R_{\mathrm{L}})$ defined in Eq. (131) does not depend on R_{L}, within this approximation. Moreover, from Eqs. (132) and (133) we see that we have the order of magnitude estimate:

$$\tilde{\epsilon} \sim \epsilon \left(\int \mathrm{d}\boldsymbol{k}\,P_\phi(k)\right)^{1/2} = \epsilon\,\sqrt{\langle\phi^2\rangle}. \tag{137}$$

Going back to the definition (105) of the initial conditions we see that we can write the primordial perturbations of the gravitational potential as:

$$\Phi(\boldsymbol{x}) = \phi(\boldsymbol{x}) + \mathcal{O}(1)\,\tilde{\epsilon}\,\frac{\phi(\boldsymbol{x})^2 - \langle\phi^2\rangle}{\sqrt{\langle\phi^2\rangle}} \tag{138}$$

where $\mathcal{O}(1)$ stands for a numerical factor of order unity. It is clear from this expression that for a scale-invariant primordial power-spectrum the initial conditions are close to Gaussian if $\tilde{\epsilon} \ll 1$. Hence $\tilde{\epsilon}$ is the relevant dimensionless parameter which describes the amplitude of the deviations from Gaussianity and the usual inflationary scenarios lead to $\tilde{\epsilon} \ll 1$.

Here it is interesting to consider the linear regime, that is the statistics of the linear density field. Then, we have $\mathcal{G}(\tau) = -\tau$ and $\tau = y = -\delta_{\mathrm{L},R} = -\delta_R$, so that $\varphi_0(y) = -y^2/2$. This yields from Eq. (129):

$$\psi_{\mathrm{L}}(y) = \mathrm{e}^{y^2\sigma_0^2/2}\left[1 - \tilde{\epsilon}\,y^3\sigma_0^3\right] \tag{139}$$

where the subscript "L" refers to the "linearly evolved" density field. Then, from the expansion (4) we obtain for the third-order moment of the linear density field $\langle\delta_{\mathrm{L},R}^3\rangle$ and for the skewness D_3:

$$\langle\delta_{\mathrm{L},R}^2\rangle = \sigma_0^2, \quad \langle\delta_{\mathrm{L},R}^3\rangle = 6\tilde{\epsilon}\sigma_0^3, \quad D_3 \equiv \frac{\langle\delta_{\mathrm{L},R}^3\rangle}{\langle\delta_{\mathrm{L},R}^2\rangle^{3/2}} = 6\tilde{\epsilon} \tag{140}$$

at first-order in $\tilde{\epsilon}$. Thus, we see that the parameter $\tilde{\epsilon}$ is directly related to the skewness of the linear density field on the scale of interest, due to the slight non-Gaussianity of the primordial density field.
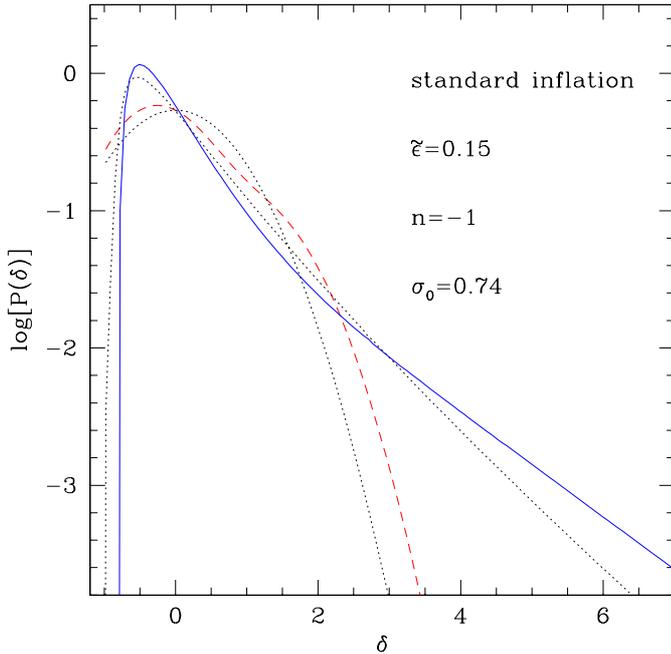
Fig. 3. The pdf $\mathcal{P}(\delta_R)$ for $n = -1$ and $\sigma_0 = 0.74$, for slightly non-Gaussian initial conditions: $\tilde{\epsilon} = 0.15$. The solid line shows the prediction of Eq. (141) for the non-linear pdf $\mathcal{P}(\delta_R)$ while the dashed-curve shows the pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly evolved density field from Eq. (143). The dotted lines display the results obtained with $\tilde{\epsilon} = 0$, i.e. for Gaussian initial conditions.

5.4. The pdf of the density contrast

From the rescaled Laplace transform $\overline{\psi}(y)$ obtained in Eq. (129) we can derive the pdf $\mathcal{P}(\delta_R)$, using the inverse Laplace transform (3). This yields:

$$\mathcal{P}(\delta_R) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}y}{2\pi i \sigma_0^2(R)} \, e^{[y\delta_R - \varphi_0(y)]/\sigma_0^2} \left[1 - \tilde{\epsilon} \, \frac{\tau^3}{\sigma_0^3}\right] (141)$$

where the function $\tau(y)$ is given by Eq. (128). The pdf $\mathcal{P}_L(\delta_{L,R})$ of the linearly evolved density field is obtained from Eq. (141) by using $\varphi_0(y) = -y^2/2$ and $\tau(y) = y$ which gives:

$$\mathcal{P}_L(\delta_{L,R}) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}y}{2\pi i \sigma_0^2} \, e^{[y\delta_{L,R} + y^2/2]/\sigma_0^2} \left[1 - \tilde{\epsilon} \, \frac{y^3}{\sigma_0^3}\right] \cdot (142)$$

Then we can perform the Gaussian integration over y in Eq. (142) which yields (see Gradshteyn & Ryzhik 1965, Sect. 3.462.4):

$$\mathcal{P}_L(\delta_{L,R}) = \frac{e^{-\delta_{L,R}^2/(2\sigma_0^2)}}{\sqrt{2\pi}\sigma_0(R)} \left[1 + \tilde{\epsilon} \, 2^{-3/2} \, H_3\left(\frac{\delta_{L,R}}{\sqrt{2}\sigma_0}\right)\right] (143)$$

where $H_3(x)$ is the Hermite polynomial of order 3 defined by:

$$H_3(x) \equiv 8x^3 - 12x. \tag{144}$$

Finally, we display in Fig. 3 our results for slightly non-Gaussian initial conditions: $\tilde{\epsilon} = 0.15$. In order to compare

the pdf with the case of Gaussian initial conditions we also show the results obtained for $\tilde{\epsilon} = 0$ (dotted curves). Let us recall that Eq. (142) for the pdf of the linearly evolved density field is exact to first-order in $\tilde{\epsilon}$. In agreement with Eq. (140) we can check that $\tilde{\epsilon} > 0$ leads to a small positive skewness for the linear density field. In particular, the high-density tail of the pdf is slightly enhanced. We can see that this feature remains valid for the non-linear pdf $\mathcal{P}(\delta_R)$. However, as gravitational clustering proceeds the deviations from Gaussianity become dominated by the non-linear dynamics over the slight primordial non-Gaussianity. More precisely, we can obtain the low-order moments of the density field through the expansion (4). At the lowest order in y we have $\tau = y + \mathcal{O}(y^2)$ and $\varphi_0(y) = -y^2/2! + S_3 y^3/3! + \mathcal{O}(y^4)$ where the parameter S_3 is given by the standard result (e.g., Bernardeau 1994a):

$$S_3 = \frac{34}{7} - (n + 3). \tag{145}$$

Then, we obtain from Eq. (129) the expansion:

$$\psi(y) = 1 + \frac{1}{2!}y^2\sigma_0^2 - \frac{1}{3!}S_3 y^3 \sigma_0^4 - \tilde{\epsilon}y^3\sigma_0^3 + \mathcal{O}(y^4). \tag{146}$$

This yields for the lowest order moments:

$$\langle \delta_R^2 \rangle = \sigma_0^2 \, , \, \langle \delta_R^3 \rangle = S_3 \, \sigma_0^4 + 6 \, \tilde{\epsilon} \, \sigma_0^3 \tag{147}$$

and for the skewness D_3:

$$D_3 \equiv \frac{\langle \delta_R^3 \rangle}{\langle \delta_R^2 \rangle^{3/2}} = 6 \, \tilde{\epsilon} + S_3 \, \sigma_0. \tag{148}$$

This relation clearly shows that the relative importance of the primordial deviation from Gaussianity is greater at earlier times before gravitational clustering builds up, as could be expected. Note that in Fig. 3 we used the generating function $\varphi_0(y)$ defined by the branch of Eq. (128) which is regular at the origin $y = 0$. Since here we considered the case $n = -1$ this function actually shows a branch cut along the real negative axis for $y < y_s$ (with $y_s < 0$). Then, for large density contrasts one needs to take into account the second branch of $\varphi_0(y)$ defined over the range $y_s < y < 0$ which is singular at the origin. However, as shown in Fig. 6 in Paper II this is irrelevant for the case displayed in Fig. 3 for $\delta_R < 7$. We refer the reader to Paper II for a detailed discussion of this point. Of course, for the non-Gaussian model we study here one simply needs to follow the procedure outlined in Paper II for Gaussian initial conditions and to add to the relevant expressions the term in the brackets in Eq. (141) which describes the effects due to the small primordial deviation from Gaussianity (to first order in $\tilde{\epsilon}$).

6. Conclusion

Thus, using a non-perturbative method developed in a previous work (Paper II), we have described in details how to obtain the pdf $\mathcal{P}(\delta_R)$ of the density contrast within

spherical cells in the quasi-linear regime for three specific non-Gaussian models.

The first case is a straightforward generalization of the Gaussian scenario and it can be seen as a phenomenological description of a density field where the tails of the linear pdf are of the form $\mathcal{P}(\delta_\mathrm{L}) \sim \mathrm{e}^{-|\delta_\mathrm{L}|^\alpha}$, where α is no longer required to be equal to two (as in the Gaussian case). Then, we have shown that the derivation of the pdf presented for the Gaussian case can be directly extended to this model. This provides again exact results for the pdf in the quasi-linear limit.

The second scenario is a physically motivated model of isocurvature cold dark matter presented in Peebles (1999a). It arises from an inflationary scenario with three scalar fields. This case is slightly more difficult as one needs to adapt the method to this specific model. Moreover, in order to get simple analytical results one must introduce a simple approximation (which is not related to the gravitational dynamics but to the non-Gaussian properties of the initial conditions). However, we have shown that even with this approximation we get good results for the linear pdf $\mathcal{P}_\mathrm{L}(\delta_{\mathrm{L},R})$ for $\delta_{\mathrm{L},R} \gtrsim 0$ by comparison with numerical simulations. For lack of data we could not check our prediction for the non-linear pdf $\mathcal{P}(\delta_R)$ but we can expect a good agreement of the same accuracy.

Finally, the third scenario corresponds to the small primordial deviations from Gaussianity which arise in standard slow-roll inflation. We obtained exact results for the pdf of the density field in the quasi-linear limit, to first-order over the primordial deviations from Gaussianity.

This study shows that our approach is powerful enough to be applied to a large variety of initial conditions. Note that our predictions for the linear pdf $\mathcal{P}_\mathrm{L}(\delta_{\mathrm{L},R})$ for such non-Gaussian scenarios may be used to estimate the mass function of just-collapsed objects, using a straightforward extension of the Press-Schechter recipe. Note that we shall discuss the Press-Schechter prescription in the light of the formalism developed in Paper II and used in this work in a companion article (Paper IV).

To conclude, we note that our approach being non-perturbative it can in principle be applied to the non-linear regime. Indeed, it does not rely either on the hydrodynamical description. We shall present a study of this non-linear regime in a future work, see Paper IV, for Gaussian initial conditions. However, it is clear that this can be extended to the non-Gaussian models described here.

Appendix A: Isocurvature scenario. Reduction to ordinary integrals

In this appendix we show how to derive the expression (79) from Eq. (58). We first consider the linear regime where Eq. (79) is exact. The path-integral (58) involves the functional measure $[\mathrm{d}\delta_\mathrm{L}(\boldsymbol{x})]$. The latter can be defined from a discretization procedure (i.e. a spatial grid for the

coordinate $\boldsymbol{x}$ with an infinitesimal spacing) but an equivalent formulation (e.g., Zinn-Justin 1989) is to expand the function $\delta_\mathrm{L}(\boldsymbol{x})$ on a complete set of real orthonormal functions f_q (in the Hilbert space $\mathcal{L}^2$):

$$\delta_\mathrm{L}(\boldsymbol{x}) = \sum_{q=0}^{\infty} a_q \, f_q(\boldsymbol{x}). \tag{A.1}$$

The vectors f_q obey the relation:

$$\int \frac{\mathrm{d}\boldsymbol{x}}{V} \, f_q(\boldsymbol{x}) f_r(\boldsymbol{x}) = \delta_{q,r} \tag{A.2}$$

which expresses the fact that they form an orthonormal basis. Here $\delta_{q,r}$ is the usual Kronecker symbol. We introduced a factor $1/V$ in the scalar product (A.2) so that the functions f_q are dimensionless, but this is not essential (here V is a constant). Then, the functional measure $[\mathrm{d}\delta_\mathrm{L}(\boldsymbol{x})]$ can be defined as (e.g., Zinn-Justin 1989):

$$[\mathrm{d}\delta_\mathrm{L}(\boldsymbol{x})] \equiv \mathcal{N} \prod_{q=0}^{\infty} \mathrm{d}a_q \tag{A.3}$$

where $\mathcal{N}$ is a normalization constant. In order to get a discrete basis in Eq. (A.1) we restricted the density field to a large finite volume $\mathcal{V}$. However, this is not essential: we choose $\mathcal{V}$ to be much larger than any relevant length scale and we can eventually take the limit $\mathcal{V} \to \infty$. Next, we can choose for the first basis vector f_0 the top-hat of radius R:

$$f_0(\boldsymbol{x}) \equiv \theta(x < R) \tag{A.4}$$

where $\theta(x < R)$ is the usual top-hat with obvious notations. Then, from Eqs. (A.1) and (A.2) we have:

$$a_0 = \int \frac{\mathrm{d}\boldsymbol{x}}{V} \, f_0(\boldsymbol{x}) \delta_\mathrm{L}(\boldsymbol{x}) = \delta_{\mathrm{L},R}. \tag{A.5}$$

We can expand the auxiliary field $\lambda(\boldsymbol{x})$ on the same basis:

$$\lambda(\boldsymbol{x}) = \sum_{q=0}^{\infty} b_q \, f_q(\boldsymbol{x}) \tag{A.6}$$

so that the path-integral (58) now writes:

$$\psi_\mathrm{L}(y) = \int_{-\infty}^{\infty} \prod_{q=0}^{\infty} \mathrm{d}a_q \prod_{r=0}^{\infty} \mathrm{d}b_r \, \mathrm{e}^{-ya_0 + iV \sum_{q=0}^{\infty} a_q b_q}$$
$$\times \, \mathrm{e}^{i\langle\phi^2\rangle(\lambda.1) - \frac{1}{2}\mathrm{Tr}\ln(1 + 2i\Lambda\Delta_\phi)} \tag{A.7}$$

where the subscript "L" refers to the "linearly evolved" density field (so that $\delta_R = \delta_{\mathrm{L},R} = a_0$) and we did not write the normalization constant of the integrals. Then, the integration over the variables a_q with $q \geq 1$ yields the Dirac functions $\delta_\mathrm{D}(b_q V)$. The integration over b_r is now straightforward for $r \geq 1$ and we obtain:

$$\psi_\mathrm{L}(y) = \int_{-\infty}^{\infty} \mathrm{d}a_0 \mathrm{d}b_0 \, \mathrm{e}^{-ya_0 + iVa_0b_0 + i\langle\phi^2\rangle b_0 V}$$
$$\times \, \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\ln(1 + 2i\Lambda\Delta_\phi)} \tag{A.8}$$

where the matrix Λ is obtained from $\lambda(\boldsymbol{x}) = b_0\theta(x < R)$. Then, using $a_0 = \delta_{L,R}$ from Eq. (A.5) and making the change of variable $\lambda_0 = b_0 V$ we obtain Eq. (79) in the linear regime (i.e. $\delta_{L,R_L} = \delta_{L,R} = \delta_R$ and $R_L = R$).

Now, we investigate how the previous derivation is modified when we study the non-linear density field $\delta(\boldsymbol{x})$. We can again define the functional measure $[\mathrm{d}\delta_L(\boldsymbol{x})]$ (and $[\mathrm{d}\lambda(\boldsymbol{x})]$) by Eq. (A.3). However, we now introduce the new variable δ_{L,R_L}, which also defines $\delta_R \equiv \mathcal{F}(\delta_{L,R_L})$ and $R_L \equiv R(1+\delta_R)^{1/3}$. Then, we define the functions $f'_q(\boldsymbol{x})$ by:

$$f'_q(\boldsymbol{x}) \equiv \sqrt{\frac{V}{V_L}}\, f_q\left(\frac{R}{R_L}\,\boldsymbol{x}\right) \tag{A.9}$$

so that we still have the orthonormalization property:

$$\int \frac{\mathrm{d}\boldsymbol{x}}{V}\, f'_q(\boldsymbol{x}) f'_r(\boldsymbol{x}) = \delta_{q,r}. \tag{A.10}$$

Then, we write the linear density field $\delta_L(\boldsymbol{x})$ as:

$$\delta_L(\boldsymbol{x}) = \sum_{q=0}^{\infty} a'_q\, f'_q(\boldsymbol{x}) \tag{A.11}$$

which also implies:

$$a'_0 = \sqrt{\frac{V_L}{V}}\, \delta_{L,R_L}. \tag{A.12}$$

This yields:

$$\delta_L(\boldsymbol{x}) = \delta_{L,R_L}\theta(x < R_L) + \sum_{q=1}^{\infty} a'_q\, f'_q(\boldsymbol{x}) \tag{A.13}$$

while we write the auxiliary field $\lambda(\boldsymbol{x})$ as:

$$\lambda(\boldsymbol{x}) = \sum_{q=0}^{\infty} b'_q\, f'_q(\boldsymbol{x}). \tag{A.14}$$

Then, the path-integral (58) now writes:

$$\psi(y) = \int_{-\infty}^{\infty} \mathrm{d}\delta_{L,R_L} \prod_{q=1}^{\infty} \mathrm{d}a'_q \prod_{r=0}^{\infty} \mathrm{d}b'_r\, J_a J_b\, \mathrm{e}^{-y\delta_R}$$
$$\times\, \mathrm{e}^{iV\sum_{q=0}^{\infty} a'_q b'_q + i\langle\phi^2\rangle(\lambda.1) - \frac{1}{2}\mathrm{Tr}\ln(1+2i\Lambda\Delta_\phi)} \tag{A.15}$$

where we made the changes of variables $\{a_q\}_{q\geq 0} \to \{\delta_{L,R_L}, \{a'_q\}_{q\geq 1}\}$ and $\{b_q\}_{q\geq 0} \to \{b'_q\}_{q\geq 0}$. In Eq. (A.15) the Jacobians J_a and J_b of these transformations are given by the determinants:

$$J_a \equiv \left|\mathrm{Det}\left[\frac{\partial a_i}{\partial\delta_{L,R_L}}, \frac{\partial a_i}{\partial a'_j}\right]\right|,\ J_b \equiv \left|\mathrm{Det}\left[\frac{\partial b_i}{\partial b'_j}\right]\right|. \tag{A.16}$$

These Jacobians do not depend on the amplitude $\sigma(R)$ of the density fluctuations. On the other hand, the quasi-linear regime corresponds to the limit $\sigma \to 0$ for a fixed finite y (which leads to finite δ_{L,R_L} and δ_R, as shown in the Gaussian case in Paper II). In this limit, the generating function $\psi(y)$ is governed by the exponent in Eq. (A.15) which depends on σ and sets the cutoffs of

order σ for the density contrast δ_R. Then, we can neglect the Jacobians in Eq. (A.15): they do not contribute to $\ln(\psi)$ at leading order for $\sigma \to 0$ (however, they give a σ-independent prefactor which appears in ψ, see the discussion below). Moreover, if we make the approximation $\delta_R[\delta_L] \simeq \mathcal{F}(\delta_{L,R_L})$ we can again integrate over a'_q for $q' \geq 1$, which gives the Dirac functions $\delta_D(b'_q V)$, and we eventually recover Eq. (79).

Let us now discuss the approximation $\delta_R[\delta_L] \simeq \mathcal{F}(\delta_{L,R_L})$ used above. The point is that in the limit $\sigma \to 0$ the path-integral (58) and the ordinary integrals (A.15) are dominated by the saddle-points of the exponent. As discussed in Sect. 4.3, because of the spherical symmetry of the physics we investigate there exist some spherical saddle-points, of the form (70). Then, if there are no other saddle-points (or if some other saddle-points exist but they yield a lower contribution to integral, which will vanish in the limit $\sigma \to 0$) the path-integral is given at leading-order by its value for these spherical saddle-points. Note that the case of Gaussian initial conditions studied in Paper II exhibits the same behaviour in a simpler manner. Indeed, in this case there exists only one spherical saddle-point for both the linear and non-linear generating functions $\psi_L(y)$ and $\psi(y)$. Then, in order to keep only the leading order for $\sigma \to 0$ one defines the rescaled generating function $\varphi(y)$ by:

$$\psi(y) = \mathrm{e}^{-\varphi(y\sigma^2)/\sigma^2} \tag{A.17}$$

as in Eq. (19), which can be expressed through the path-integral:

$$\mathrm{e}^{-\varphi(y)/\sigma^2} = \int [\mathrm{d}\delta_L(\boldsymbol{x})]\, \mathrm{e}^{-S[\delta_L]/\sigma^2} \tag{A.18}$$

in a fashion similar to Eq. (20). Next, in the quasi-linear limit $\sigma \to 0$ the path-integral in the r.h.s. in Eq. (A.18) is dominated by one saddle-point and taking the logarithm of both sides in Eq. (A.18) one obtains in this limit:

$$\varphi(y) = \min_{\delta_L(\boldsymbol{x})} S[\delta_L] \tag{A.19}$$

as described in Paper II (see also Sect. 3.2). Going back to $\psi(y)$ this means that in the quasi-linear limit the generating function $\psi(y)$ is given by the maximum of the exponent which appears in the relevant path-integral. This corresponds to the procedure we described above for the isocurvature CDM scenario in order to get Eq. (79). However, the difference is that in this non-Gaussian case there exists an infinite number of spherical saddle-points with respect to $\delta_L(\boldsymbol{x})$. Therefore, at leading order the generating function $\psi(y)$ is not given by a unique contribution but rather by the sum of all contributions due to these saddle-points which are parameterized by the variables δ_{L,R_L} and λ_0.

Finally, we note that the minimum of the action $S[\delta_L]$, or the value of the exponent in Eq. (A.15), only provides the leading order contribution to $\ln(\psi)$ in the limit $\sigma \to 0$. This means that it yields the exact exponential σ-dependent term of the pdf $\mathcal{P}(\delta_R)$ which describes the cutoffs of the pdf for density contrasts which are large

relative to σ. In our case, this is the factor $e^{-(1+\delta_L/\sigma_\phi^2)/2}$ which appears in Eq. (92). However, it does not give the exact multiplicative prefactor of the generating function ψ and of the pdf $\mathcal{P}(\delta_R)$. Indeed, the Jacobians J_a and J_b in Eq. (A.15) (or the Gaussian integration around the saddle-point in the Gaussian case) contribute to this term. In the case we study here, this means that Eq. (79) differs from the exact result by a multiplicative factor which may depend on δ_L and λ. Therefore, the expression (92) obtained for the pdf is correct up to a multiplicative factor which may depend on δ_R but not on σ. This point is also discussed in Sect. 4.4 where we compare Eq. (92) with the result (102) of a simple spherical model. The latter can be seen as an attempt to model this prefactor. Note that for the pdf of the linear density field we do not encounter this difficulty since, as shown above, Eq. (A.8) is actually exact.

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