

# Self-similar spherical shock solution with sustained energy injection

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**Abstract.** We present the generalization of the Sedov-Taylor self-similar strong spherical shock solution for the case of a central energy source varying in time,  $E = At^k$ , where  $A$  and  $k$  are constants. The known Sedov-Taylor solution corresponds to a particular adiabatic case of  $k = 0$  or *instant shock* with an instant energy source of the shock,  $E = A$ . The self-similar hydrodynamic flow in the nonadiabatic  $k \neq 0$  case exists only under the appropriate local entropy (energy) input which must be supported by some radiative mechanism from the central engine. The specific case of  $k = 1$  corresponds to a permanent energy injection into the shock, or *injection shock* with a central source of constant luminosity,  $L = A$ ,  $E = At$ . The generalized self-similar shock solution may be applied to astrophysical objects in which the duration of central source activity is longer than the shock expansion time, e.g. the early phase of SN explosions, strong wind from stars and young pulsars, non-steady spherical outflow from black holes and collapsing dense stellar clusters with numerous neutron star collisions.

**Key words.** hydrodynamics – shock waves – stars: winds, outflows – supernovae: general – ISM: bubbles

## 1. Introduction

The well known spherical shock solution (Sedov 1946; von Neumann 1947; Taylor 1950; Stanyukovich 1969) describes the self-similar expansion of a strong spherical shock generated by instant deposition of energy  $E = \text{const}$  by the central source in a homogeneous gas medium with density  $\rho_1 = \text{const}$ . This *instant shock* solution is commonly used in numerous astrophysics applications e.g. for the modeling of SN explosions and evolution of young SN remnants. The corresponding solution for a strong ultra-relativistic blast wave was obtained by Blandford & McKee (1976). For recent reviews of astrophysical shock models see Ostriker & McKee (1995) and Bisnovatyi-Kogan & Silich (1995) and references therein. The basic requirement for the realization of the instant shock solution is a short-duration injection of energy  $E$  into the shock. However there exist possible physical situations of permanent injection of energy into the expanding shock which we call the *injection shock*, when a central source has some time-varying luminosity  $L = L(t)$ . More exactly the instant shock solution is not applicable when the duration of energy generation by the central source  $t_s$  is comparable or exceeds the shock expansion time  $t_{sh}$ ,  $t_s \geq t_{sh}$ . The condition  $t_s \geq t_{sh}$  is typical for the early stage of a SN explosion, powerful wind from stars, and non-steady spherical accretion onto compact objects. The other example is the injection shock produced in *hidden neutrino sources* (Berezinsky & Dokuchaev 2001) by successive

multiple fireballs after numerous neutron star collisions in the dense stellar cluster in a galactic nucleus prior to its collapse into a massive black hole. Below we derive the extension of the Sedov-Taylor self-similar spherical shock solution to the case of varying in time energy injection by the central source of power form  $E = At^k$ , where  $A$  and  $k$  are constants. The notations and logistics of “Fluid Dynamics” by Landau & Lifshitz (1959, Chapter X, Sect. 106) are used in the self-similar solution derivation.

Let us consider a strong expanding spherical shock in an ideal gas with polytropic index (Poisson parameter)

$$\gamma = c_p/c_v = \text{const}, \quad (1)$$

where  $c_p$  and  $c_v$  are the gas heat capacities under constant pressure and volume respectively. All values on the forward side of the shock discontinuity surface (non-perturbed gas side) are designated by index 1, e.g.  $\rho_1$ ,  $p_1$ , and behind the discontinuity surface (shock cavity side) by index 2, e.g.  $\rho_2$ ,  $p_2$ . In a strong shock the pressure behind the shock  $p_2$  far exceeds the pressure in the non-perturbed gas  $p_1$ . The precise definition for a strong shock (determined from the shock adiabat) is  $p_2/p_1 \gg (\gamma + 1)/(\gamma - 1)$  and is similar to condition  $u_1 \gg c_s$ , where  $u_1$  is the shock expansion velocity with respect to the non-moving (non-perturbed) gas and  $c_s = (\gamma p/\rho)^{1/2}$  is the sound speed in the non-moving gas. The following relations are valid for a strong shock discontinuity:

$$v_2 = \frac{2}{\gamma + 1}u_1, \quad \rho_2 = \frac{\gamma + 1}{\gamma - 1}\rho_1, \quad p_2 = \frac{2}{\gamma + 1}\rho_1 u_1^2. \quad (2)$$

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These relations are the (outer) boundary conditions for our problem.

## 2. Self-similar solution ansatz

The ansatz for self-similar expansion of a strong spherical shock is in the observation that gas motion after the shock is determined by only two independent parameters: the initial gas density  $\rho_1 = \text{const}$  and the total shock energy  $E = \text{const}$  because for a strong shock we may put  $p_1 = 0$ . From these two parameters and from two independent variables, radius  $r$  and time  $t$  it is possible to construct the only non-dimensional combination  $r(\rho_1/Et^2)^{1/5}$  (see e.g. Landau & Lifshitz 1959; Sedov 1969; Stanyukovich 1969). As a result the gas motion reveals self-similar behaviour when different spherical parts of the gas after the shock evolve under a constant value of this non-dimensional combination. The law for the shock radius evolution would be

$$R = R(t) = \beta \left( \frac{Et^2}{\rho_1} \right)^{1/5}, \quad (3)$$

where the constant  $\beta$  itself is determined from the exact solution. The corresponding velocity of shock expansion with respect to the non-perturbed gas is  $u_1 = dR/dt$ . The gas motion behind the shock would be governed by the non-dimensional *self-similar* variable

$$\xi = \frac{r}{R(t)}, \quad (4)$$

which is the relative radius with respect to the instant shock radius  $R(t)$ . On the surface of shock discontinuity  $\xi = 1$ .

A helpful hint for finding the generalization of instant shock solution is in the following observation: the discussed self-similarity survives if a shock energy  $E$  is power-law function of time,  $E = At^k$ , with  $A = \text{const}$  and  $k = \text{const}$ . This is because the self-similar variable  $\xi$  is the power-law function of time. Putting  $E = At^k$  retains the power law dependence of the variable  $\xi$  on time. The only complication is in changing the power index in the definition of  $\xi$  from  $2/5$  to  $(2+k)/5$ . Now the case  $k = 0$  corresponds to the known instant shock solution and, for example the case  $k = 1$  would correspond to the shock with a constant luminosity of the shock source,  $A = L = \text{const}$ . In the following we find the corresponding self-similar solution of fluid equations in a general case of arbitrary  $k$ . The shock radius in general case evolves as

$$R = R(t) = \beta \left( \frac{A}{\rho_1} \right)^{1/5} t^{(2+k)/5}. \quad (5)$$

The velocity of shock expansion is

$$u_1 = \frac{dR}{dt} = \frac{(2+k)R}{5t}. \quad (6)$$

The non-dimensional self-similar variable is defined as

$$\xi = \frac{r}{R(t)} = \frac{1}{\beta} \left( \frac{\rho_1}{A} \right)^{1/5} \frac{r}{t^{(2+k)/5}}. \quad (7)$$

## 3. Fluid equations

We use the set of fluid equations for describing the shocked gas motion, as follows. The continuity equation in spherical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial r} + \frac{2\rho v}{r} = 0. \quad (8)$$

The momentum conservation (Euler) equation:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} - \frac{1}{\rho} \frac{\partial p}{\partial r} = 0. \quad (9)$$

The entropy equation:

$$\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial r} = \dot{s}, \quad (10)$$

where  $\dot{s} = \dot{s}(r, t)$  is the local entropy source (rate of local entropy production per unit mass). The self-similar solution with energy injection (at  $k \neq 0$ ) demands the existence of some non-hydrodynamic ‘‘radiative’’ mechanism for local energy (and entropy) supply from the central source into the shocked gas. In other words it is assumed that additional energy is radiatively pumped into the shock from the central source. For the realization of self-similar motion the entropy source  $\dot{s}(r, t)$  must be properly tuned. The required entropy source term is calculated from the other fluid equations by assuming self-similar motion (Eq. (25)). In the particular case of instant shock ( $k = 0$ ) the entropy source term equals zero and the gas motion is adiabatic.

The local energy equation (energy conservation law) is written as

$$\frac{\partial}{\partial t} \left( \rho \varepsilon + \rho \frac{v^2}{2} \right) = -\text{div} \left\{ \rho v \left( w + \frac{v^2}{2} \right) \right\} + \rho T \dot{s}, \quad (11)$$

where the gas internal energy density  $\varepsilon = c_v T = c_s^2 / [\gamma(\gamma - 1)]$ ,  $w = c_p T = c_s^2 / (\gamma - 1)$  is the gas enthalpy density,  $T$  is the gas temperature and  $c_s$  is the sound velocity. The integral form of the energy conservation Eq. (11) is

$$\frac{\partial}{\partial t} \int \left( \rho \varepsilon + \rho \frac{v^2}{2} \right) dV = -\oint \left\{ \rho v \left( w + \frac{v^2}{2} \right) \right\} df + \int \rho T \dot{s} dV. \quad (12)$$

From the last equation we find the rate of energy injection into the shock

$$\dot{E} = \frac{\partial}{\partial t} \int_0^R \left( \rho \varepsilon + \rho \frac{v^2}{2} \right) dV = \int_0^R \rho T \dot{s} dV, \quad (13)$$

where  $R = R(t)$  is the outer shock radius. We assume that this energy injection into the shock is supplied by the central source luminosity  $L = \dot{E} = kAt^{k-1}$  with the properly tuned local entropy source  $\dot{s} = \dot{s}(r, t)$ .

## 4. Integral of self-similar motion

The crucial step for finding the strong spherical shock self-similar solution is the guess of the integral of self-similar motion derived from the energy balance equation for a chosen sphere expanding in a self-similar manner by the law  $\xi \equiv r/R(t) = \text{const}$ . This law of motion determines according to

Eqs. (6) and (7) the corresponding radial expansion velocity of this chosen sphere:

$$v_n = \frac{d}{dt} [\xi R(t)] = u_1 \xi. \quad (14)$$

We derive the required integral of self-similar motion in full analogy with Landau & Lifshitz (1959) procedure for adiabatic ( $k = 0$ ) case. The leaking of energy through the sphere surface  $4\pi r^2$  of gas with local velocity  $v$  and density  $\rho$  during the small time interval  $\Delta t$  is

$$\rho v \left( w + \frac{v^2}{2} \right) 4\pi r^2 \Delta t. \quad (15)$$

The last equation contains the enthalpy  $w = \varepsilon + p/\rho$ , where the gas internal energy density  $\varepsilon = c_v T = c_s^2 / [\gamma(\gamma - 1)]$ , because the gas produces some work under expansion. On the other hand the volume of the sphere increases during this time interval  $\Delta t$  by the value  $4\pi r^2 \Delta t v_n$ . Inside this volume there is gas with energy

$$\rho v_n \left( \varepsilon + \frac{v^2}{2} \right) 4\pi r^2 \Delta t. \quad (16)$$

The central source pumps into this volume by a nonadiabatic process the additional energy

$$\rho v_n T \Delta s 4\pi r^2 \Delta t, \quad (17)$$

where  $\Delta s = \dot{s} \Delta t$ . This additional energy is proportional to the square of  $\Delta t$ , and  $\Delta t$  is a small time interval. From the energy balance equation

$$\rho v \left( w + \frac{v^2}{2} \right) 4\pi r^2 \Delta t = \left[ \rho v_n \left( \varepsilon + \frac{v^2}{2} \right) + \rho v_n T \Delta s \right] 4\pi r^2 \Delta t \quad (18)$$

in the thin spherical shell of radius  $r < R(t)$ , thickness  $\Delta r = v_n \Delta t$  and volume  $4\pi r^2 \Delta r$  in the limit  $\Delta t \rightarrow 0$  we obtain finally the integral of self-similar motion

$$v \left( w + \frac{v^2}{2} \right) = v_n \left( \varepsilon + \frac{v^2}{2} \right). \quad (19)$$

The left hand side and right hand side of this integral of motion is simply a different representation of the same energy flux and does not contain directly the local entropy source  $\dot{s} = \dot{s}(r, t)$ . Nevertheless this entropy source is taken into account in Eq. (19) indirectly through connections between the thermodynamic values  $w$  and  $\varepsilon$  and the entropy relation  $s = \ln(p/\rho^\gamma)$  for an ideal gas. As a result the required integral of self-similar motion (19) has the same functional form for both the adiabatic  $k = 0$  an nonadiabatic  $k \neq 0$  cases.

### 5. Solution of self-similar equations

We will find the self-similar solution for strong spherical shock with energy injection by using three equations: the continuity Eq. (8), momentum Eq. (9) and the integral of motion (19). The entropy Eq. (10) will be used for the determination of the entropy rate source  $\dot{s}$  corresponding to a suitable injection of energy from the central source.

Let us define the non-dimensional variables  $V = V(\xi)$ ,  $G = G(\xi)$ ,  $Z = Z(\xi)$  instead of gas velocity  $v$ , density  $\rho$  and sound velocity  $c_s$ :

$$v = \frac{(2+k)r}{5t} V; \quad \rho = \rho_1 G; \quad c_s^2 = \left[ \frac{(2+k)r}{5t} \right]^2 Z. \quad (20)$$

According to Eq. (2) on the shock discontinuity surface, i.e. at  $\xi = 1$ , these non-dimensional functions take the values

$$V(1) = \frac{2}{\gamma+1}; \quad G(1) = \frac{\gamma+1}{\gamma-1}; \quad Z(1) = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}. \quad (21)$$

The integral of motion from Eq. (19) in non-dimensional variables takes the form

$$Z = \frac{2 - (3 + \gamma)V + 2\gamma V^2}{V(V-1)(\gamma V-1)} \quad (22)$$

and does not depend on  $k$ . With this integral of motion the continuity Eq. (8) takes the form

$$(1-V) \frac{d \ln G}{d \ln \xi} - \frac{dV}{d \ln \xi} = 3V. \quad (23)$$

Correspondingly the Euler Eq. (9) can be written in the form

$$\frac{d \ln G}{d \ln \xi} - \frac{2 - 2(\gamma+2)V + (3+2\gamma+\gamma^2)V^2 - 2\gamma V^3}{(\gamma-1)(1-V)V^2(\gamma V-1)} \frac{dV}{d \ln \xi} = \frac{2\{5 + V[(2+k)(V-2) + \gamma(k-3)]\}}{(\gamma-1)(2+k)(1-V)V}. \quad (24)$$

Equations (22)–(24) for functions  $Z$ ,  $V$ , and  $G$  define completely the motion of the shocked gas. By using these three equations and the expression  $s = c_v \ln(p/\rho^\gamma)$  for the gas entropy we now calculate from Eq. (10) the required entropy source inside the shocked gas

$$\dot{s}(r, t) = \frac{c_v}{t} \frac{V}{1-V} (\gamma-1)k. \quad (25)$$

This entropy source is needed for the implementation of self-similar motion of the shocked gas with energy injection in accordance with the law of local energy conservation (11) or the law of energy input into the shock (13). We will find below from the exact solution that  $2/(\gamma+1) \leq V \leq 1/\gamma$ . So  $ds/dt = 0$  in the adiabatic case of  $k = 0$  and  $ds/dt > 0$  in the non-adiabatic case of  $k > 0$ . See Fig. 4 for the radial entropy rate profile for the case of  $k = 1$ . Meanwhile the entropy source becomes negative,  $\dot{s} < 0$ , at  $k < 0$ . It seems that a physically reasonable self-similar solution does not exist for the  $k < 0$  case. Resolving the system of Eqs. (24) and (23) with respect to derivatives  $dV/d \ln \xi$  and  $d \ln G/d \ln \xi$  we obtain correspondingly:

$$\frac{d \ln V}{d \ln \xi} = \frac{(\gamma V-1) \{ 10 - 2[2(2+k) - \gamma(k-3)]V + (3\gamma-1)(2+k)V^2 \}}{(2+k)(1-V) [2 - 2(\gamma+1)V + \gamma(\gamma+1)V^2]} \quad (26)$$

and

$$\frac{d \ln G}{d \ln \xi} = V \left\{ \frac{2(1-V)[1+(2\gamma-7)V+4\gamma V^2]}{(2+k)(1-V)^2[2-2(\gamma+1)V+\gamma(\gamma+1)V^2]} + \frac{k[6-8(\gamma+1)V+(7+2\gamma+5\gamma^2)V^2-4\gamma V^3]}{(2+k)(1-V)^2[2-2(\gamma+1)V+\gamma(\gamma+1)V^2]} \right\}. \quad (27)$$

Now it is possible to integrate the last two equations separately by using the boundary conditions from Eq. (21). The integration of Eq. (26) is simple but rather tedious. We find the solution for  $V(\xi)$  in parametric form:

$$\xi = \left\{ (\gamma+1)^2 \frac{10+V[V(2+k)(3\gamma-1)+2\gamma(k-3)-4(2+k)]}{2(\gamma-1)(7-\gamma+2k(3+\gamma))} \right\}^{a_4} \times \left[ \frac{(\gamma+1)(\gamma V-1)}{\gamma-1} \right]^{a_3} \times \left[ \frac{(\gamma+1)V}{2} \right]^{a_2} \times \left\{ \frac{(\sqrt{D}+A)[\sqrt{D}(\gamma+1)+\gamma^2(k-3)+5\gamma(1+k)-4(2+k)]}{(\sqrt{D}-A)[\sqrt{D}(\gamma+1)-\gamma^2(k-3)-5\gamma(1+k)+4(2+k)]} \right\}^{\frac{a_1}{\sqrt{D}}} \times \left\{ (\gamma+1)^2 \frac{10+V[V(2+k)(3\gamma-1)+2\gamma(k-3)-4(2+k)]}{2(\gamma-1)[7-\gamma+2k(3+\gamma)]} \right\}^{a_8} \times \left\{ \frac{(\sqrt{D}+A)[\sqrt{D}(\gamma+1)+\gamma^2(k-3)+5\gamma(1+k)-4(2+k)]}{(\sqrt{D}-A)[\sqrt{D}(\gamma+1)-\gamma^2(k-3)-5\gamma(1+k)+4(2+k)]} \right\}^{\frac{a_6}{\sqrt{D}}}. \quad (28)$$

in the case of  $D > 0$  and

$$\xi = \left\{ (\gamma+1)^2 \frac{10+V[V(2+k)(3\gamma-1)+2\gamma(k-3)-4(2+k)]}{2(\gamma-1)[7-\gamma+2k(3+\gamma)]} \right\}^{a_4} \times \left[ \frac{(\gamma+1)V}{2} \right]^{a_2} \left[ \frac{(\gamma+1)(\gamma V-1)}{\gamma-1} \right]^{a_3} \exp\left(\frac{2a_1}{\sqrt{D}} \arctan W\right) \quad (29)$$

in the case of  $D < 0$  respectively. In these equations the numerical constants

$$A = V(3\gamma-1)(2+k) + k(\gamma-2) - 3\gamma - 4;$$

$$D = \gamma^2(k-3)^2 + 2(2+k)(9+2k) - 2\gamma(2+k)(9+2k);$$

$$W = \frac{(3\gamma-1)(2+k)[2-V(\gamma+1)]\sqrt{-D}}{a_5\{(2+k)[(3\gamma-1)V-2]+\gamma(k-3)\}-D(\gamma+1)}; \quad (30)$$

and

$$a_1 = \frac{(\gamma-2)[3(12-7\gamma+13\gamma^2)+(2\gamma+1)(3\gamma-1)k^2]}{10(2\gamma+1)(3\gamma-1)} + \frac{k(33\gamma+16\gamma^2+19\gamma^3-32)}{10(2\gamma+1)(3\gamma-1)};$$

$$a_2 = -\frac{(2+k)}{5};$$

$$a_3 = \frac{\gamma-1}{2\gamma+1};$$

$$a_4 = \frac{\gamma[7+k+\gamma(6k-13)]-12-k}{10(2\gamma+1)(3\gamma-1)};$$

$$a_5 = 4(2+k) - \gamma[5-3\gamma+(5+\gamma)k]. \quad (31)$$

From Eq. (28) and (29) it follows that  $V(0) = 1/\gamma$  is independent of  $k$ .

Similarly we find the solution for  $G(\xi)$  in parametric form by integrating of Eq. (27).

**Table 1.** Numerical values of constant  $\beta(\gamma, k)$  for shocks with different  $\gamma$  and  $k = 0$  (instant shock with  $E = \text{const}$ ),  $k = 1$  (injection shock with  $L = \text{const}$ ) and  $k = 0$  (Shock with  $E = At^2$ ).

$\gamma =$	5/3	7/5	4/3
$k = 0$	1.152	1.033	0.994
$k = 1$	0.929	0.826	0.793
$k = 2$	0.368	0.288	0.271

Case  $D > 0$ :

$$G = \frac{1}{1-V} \left[ \frac{(\gamma+1)(\gamma V-1)}{\gamma-1} \right]^{a_7} \times \left\{ (\gamma+1)^2 \frac{10+V[V(2+k)(3\gamma-1)+2\gamma(k-3)-4(2+k)]}{2(\gamma-1)[7-\gamma+2k(3+\gamma)]} \right\}^{a_8} \times \left\{ \frac{(\sqrt{D}+A)[\sqrt{D}(\gamma+1)+\gamma^2(k-3)+5\gamma(1+k)-4(2+k)]}{(\sqrt{D}-A)[\sqrt{D}(\gamma+1)-\gamma^2(k-3)-5\gamma(1+k)+4(2+k)]} \right\}^{\frac{a_6}{\sqrt{D}}}. \quad (32)$$

Case  $D < 0$ :

$$G = \frac{1}{1-V} \left[ \frac{(\gamma+1)(\gamma V-1)}{\gamma-1} \right]^{a_7} \exp\left(\frac{2a_6}{\sqrt{-D}} \arctan W\right) \times \left\{ (\gamma+1)^2 \frac{10+V[V(2+k)(3\gamma-1)+2\gamma(k-3)-4(2+k)]}{2(\gamma-1)[7-\gamma+2k(3+\gamma)]} \right\}^{a_8}, \quad (33)$$

where

$$a_6 = \frac{3(\gamma+3)\{2+k-\gamma[2+\gamma(k-3)+k]\}}{(3\gamma-1)(2\gamma+1)};$$

$$a_7 = \frac{3}{2\gamma+1};$$

$$a_8 = \frac{3(\gamma^2+1)}{(2\gamma+1)(3\gamma-1)}. \quad (34)$$

### 6. The calculation of constant $\beta$ from the energy integral

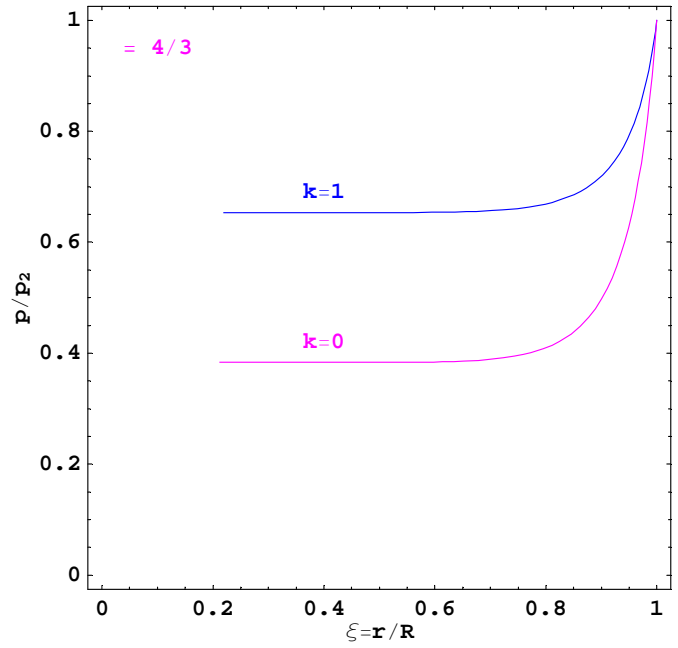
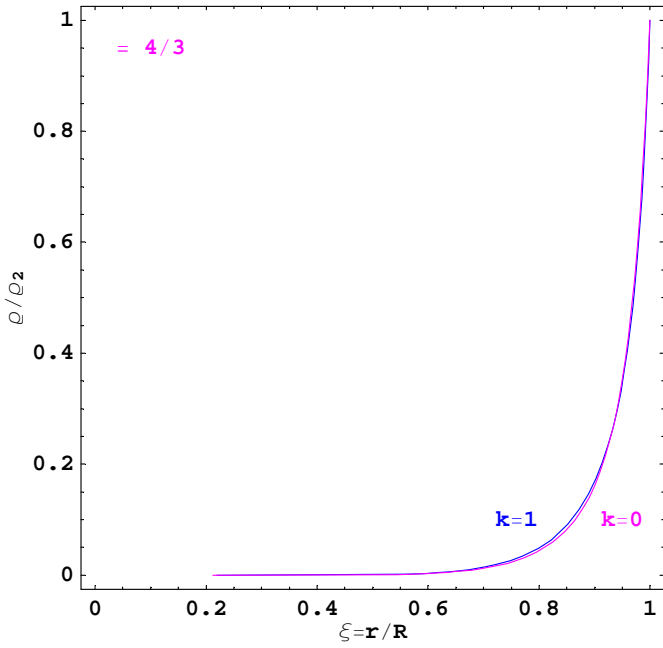
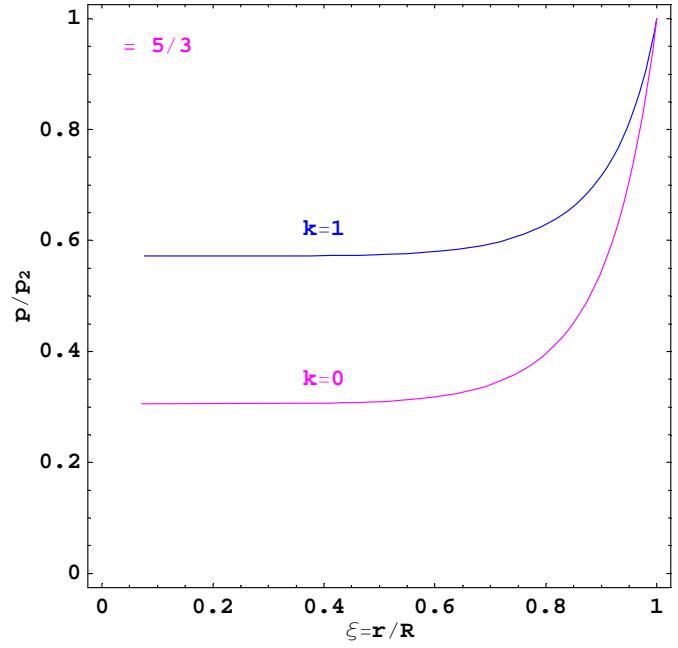
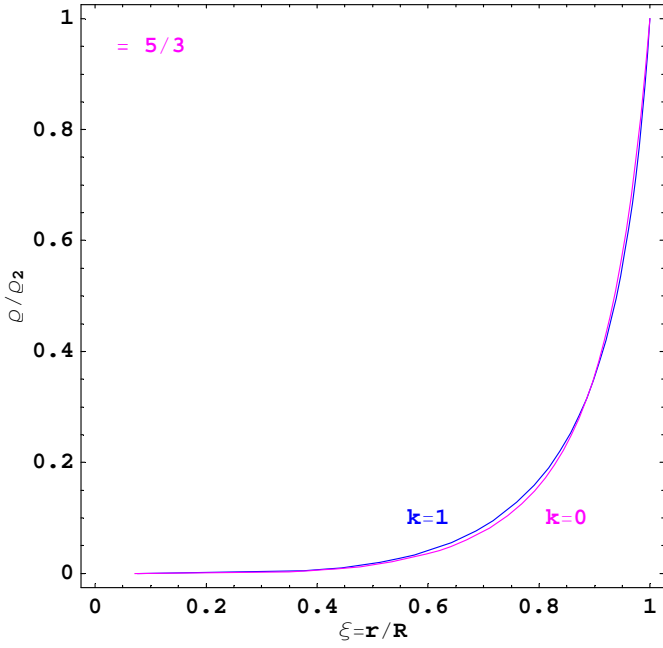
The constant  $\beta$  which appears in a self-similar variable  $\xi$  can be found from the shock total energy:

$$E = \int_0^R \left( \rho \frac{v^2}{2} + \frac{p}{\gamma-1} \right) 4\pi r^2 dr. \quad (35)$$

In non-dimensional variables this can be written as

$$\beta^5 \frac{4\pi}{25} (2+k)^2 \int_0^1 G \left( \frac{V^2}{2} + \frac{Z}{\gamma(\gamma-1)} \right) \xi^4 d\xi = 1. \quad (36)$$

By putting into this integral the calculated functions  $Z, V, G$  from (22), (28) and (32) we find after numerical integration the required  $\beta = \beta(\gamma, k)$ . This calculation of a numerical constant  $\beta$  completes the finding of generalized self-similar shock solution. See Table 1 for some examples of numerical values of  $\beta = \beta(\gamma, k)$ .



**Fig. 1.** Shock density profile  $\rho/\rho_2 = G(\xi)(\gamma - 1)/(\gamma + 1)$  depending on the relative radius of the shock  $\xi = r/R(t)$  for  $\gamma = 5/3$  and  $4/3$ ;  $k = 0$  corresponds to Sedov-Taylor case with the shock energy  $E = \text{const}$ ;  $k = 1$  corresponds to the shock with a luminosity of the central source  $L = \text{const}$ .

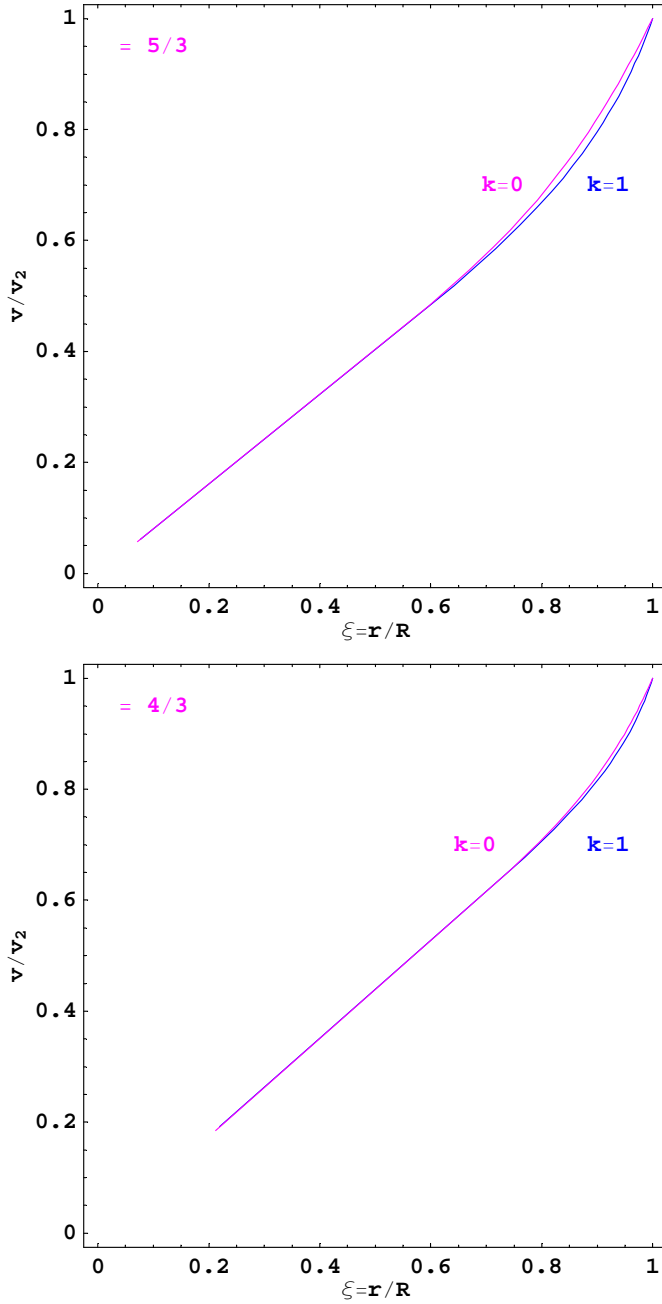
**Fig. 2.** Shock pressure profile  $p(r)/p_2 = (\frac{\gamma+1}{2\gamma})\xi G(\xi)Z(\xi)$  depending on the relative radius of the shock  $\xi = r/R(t)$  for  $\gamma = 5/3$  and  $4/3$ .

### 7. Shock with permanent energy injection

The particular case of the generalized shock solution for astrophysical application is a permanent energy *injection shock* produced by a continuous pumping of energy into the shock from the central source of constant luminosity,  $L = \text{const}$ . It corresponds to the case of  $k = 1$  and  $E = Lt \propto t$  in contrast to the usually applied Sedov-Taylor shock with  $E = \text{const}$ , which is produced by a instant implosion of energy into the shock. The possible astrophysical implications of *injection shock* solutions with  $k \neq 0$ , that is with a central energy source varying in time,

are the early phase of SN explosion, rarefied bubbles in the interstellar medium after SN explosions, strong wind from stars and young pulsars, non-steady spherical outflow from accreting black holes and dense stellar clusters near collapse with frequent neutron star collisions. Different analytical and numerical approaches were applied by Falle (1975), Castor et al. (1975), Weaver et al. (1977) to the modeling of permanent energy injection shocks in the case of stellar winds, and on the interstellar medium and interstellar bubbles.

The external radius of the expanding  $k = 1$  injection shock evolves with time according to Eq. (5) as  $R(t) = \beta(A/\rho_1)^{1/5} t^{(2+k)/5} \propto t^{3/5}$ , where  $L = A$  is the luminosity of the central source and  $\rho_1 = \text{const}$  is the density of the ambient

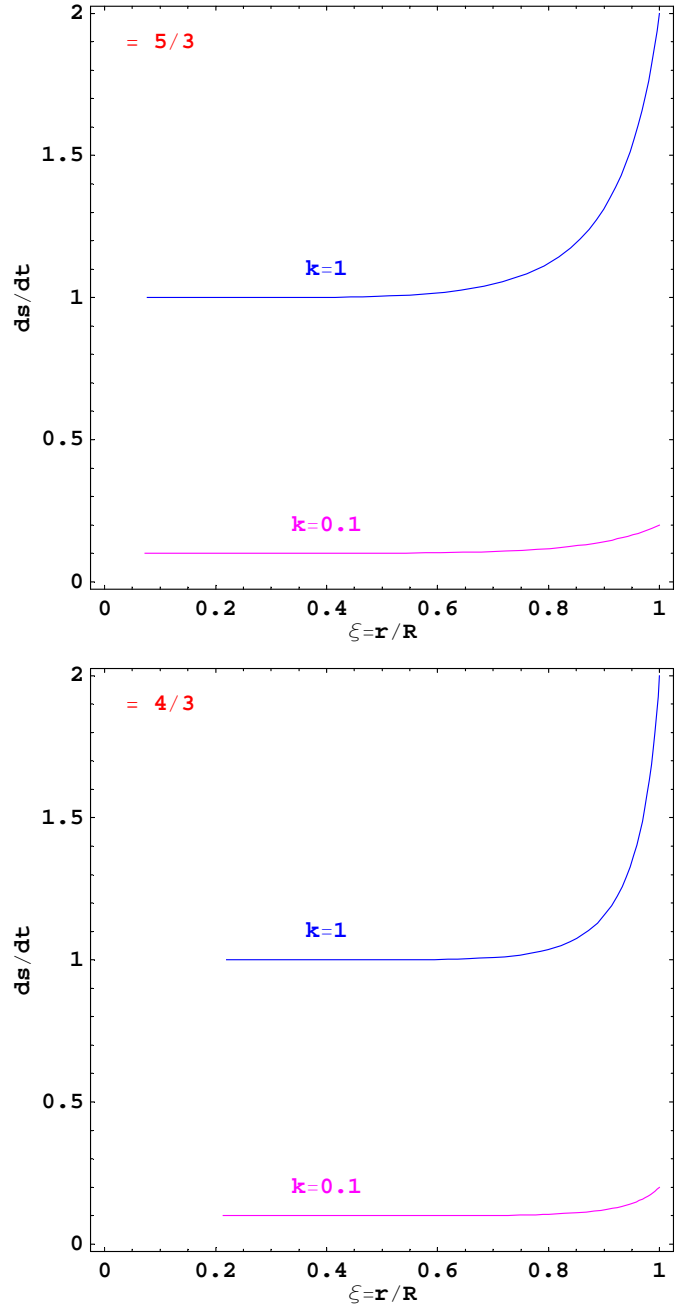


**Fig. 3.** Shock velocity profile  $v/v_2 = \xi V(\xi)(\gamma + 1)/2$  depending on the relative radius of the shock  $\xi = r/R(t)$  for  $\gamma = 5/3$  and  $4/3$ .

gas medium. So the injection shock expands faster than the instant Sedov-Taylor shock ( $k = 0$ ,  $R \propto t^{2/5}$ ). This is because of a constant pumping of energy into the shock,  $L = \text{const}$ . The corresponding velocity of injection shock expansion is  $u_1(t) = dR/dt = (R/t)(k + 2)/5 \propto t^{-2/5}$ . From the last two equations it is very clear the physical meaning and the difference between the instant shock ( $k = 0$ ) and injection shock ( $k = 1$ ) cases respectively is very clear:

$$E = \left[ (5/2)^2 \beta(\gamma, 0)^{-5} \right] R^3 (\rho u^2) = \text{const}; \quad (37)$$

$$L = \left[ (5/3)^3 \beta(\gamma, 1)^{-5} \right] R^2 u (\rho u^2) = \text{const}. \quad (38)$$



**Fig. 4.** Relative entropy rate  $(ds/dt)/(c_v/t)$  profile, where  $ds/dt$  from Eq. (25), depending on the relative radius of the shock  $\xi = r/R(t)$  for  $\gamma = 4/3$  and  $5/3$ , and  $k = 0.1$  and  $1$ .

In these expressions  $R^3 \sim V$  is shock volume and  $R^2 \sim S$  is the shock surface. At  $k = 0$  we have the expansion law for instant shock which corresponds to *constant energy* carried by the swept out gas. The expansion law for a  $k = 1$  injection shock corresponds to *constant energy flux* (or constant source luminosity) carried by the swept out gas.

The discussed strong shock solution is valid only in the region where the shock expansion velocity  $c_s \ll u(R) \ll c$ , where  $c_s$  is the sound speed in the ambient gas. The expanding strong shock becomes weak and disappears when its expansion velocity drops below the the sound speed  $c_s$ . The maximum

radius of the expanding strong shock  $R_{\text{sh}}$  is obtained from the equality  $u(R_{\text{sh}}) = c_s$  by using Eqs. (5) and (6):

$$R_{\text{sh}} = \left[ \left( \frac{2+k}{5c_s} \right)^{2+k} \beta^5 \frac{A}{\rho_1} \right]^{\frac{1}{3-k}}. \quad (39)$$

The maximum time of a strong shock expansion is

$$t_{\text{sh}} = \left[ \left( \frac{2+k}{5c_s} \beta \right)^5 \frac{A}{\rho_1} \right]^{\frac{1}{3-k}}. \quad (40)$$

The minimal radius of Newtonian motion of the shocked gas, which is called the Sedov length  $l_S$ , is defined by equality  $u(l_S) = c$ . The corresponding expression for the Sedov length,  $l_S = R_{\text{sh}}(c_s/c)^{(2+k)/(3-k)} \ll R_{\text{sh}}$ , is reproduced from Eq. (39) by substituting  $c$  for  $c_s$ . So the region of applicability of the strong shock solution is  $l_S \ll r \ll R_{\text{sh}}$ .

## 8. Conclusion

The derived solution is the generalization of the Sedov-Taylor self-similar strong spherical shock solution for the case of an energy injection from the central source of form  $E = At^k$ , where  $A$  and  $k$  are constants. The power-law ansatz  $E = At^k$  only is enough for deriving the scaling law for shock radius (5) and shock expansion velocity (6) accurate to within the numerical constant  $\beta(\gamma, k) \sim 1$  without knowing the exact solution. The numerical value of this constant (see Table 1) can be calculated from Eq. (36) only after the complete solving of the self-similar problem. The special case of  $k = 0$  corresponds to the known Sedov-Taylor solution, while the case  $k = 1$  corresponds to permanent energy injection into the shock by a central source of constant luminosity. The cases with  $k < -1$  seem to be nonphysical due to the total energy divergence at  $t \rightarrow 0$ .

The self-similar hydrodynamic flow in the nonadiabatic  $k \neq 0$  case exists only under the self-consistency condition (25) for the local entropy input. In other words the self-similar behavior of an expanding shock in the nonadiabatic  $k \neq 0$  case is realised only under the appropriate tuning of local entropy (energy) source according to Eq. (25). This is the auxiliary physical condition which supposes some radiative mechanism for sustained energy supply from the central source into the

shocked gas (which depends on the detailed properties of the central engine, radiative transfer, gas composition etc.). It can be seen from Figs. 1 and 4 that the main part of the energy is injected near the outer boundary of the shock at  $\xi \geq 0.8$ , i.e. at the same place where the shocked gas is mainly gathered. The similarities of the profiles for density and entropy rate are in favor of the principal realization of the required tuning of the local energy injection mechanism if the central source radiation absorption would be proportional to the gas density.

The self-similar shock solution with energy injection may be applied to the modeling of astrophysical objects in which duration of central source activity is longer than shock expansion time, such as the early phase of SN explosion, strong wind from stars and young pulsars, non-steady spherical outflow from black holes and collapsing dense stellar clusters with numerous neutron star collisions.

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