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Stellar evolution with rotation and magnetic fields

I. The relative importance of rotational and magnetic effects

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Abstract. We compare the current effects of rotation in stellar evolution to those of the magnetic field created by the Tayler instability. In stellar regions, where a magnetic field can be generated by the dynamo due to differential rotation (Spruit 2002), we find that the growth rate of the magnetic instability is much faster than for the thermal instability. Thus, meridional circulation is small with respect to the magnetic fields, both for the transport of angular momentum and of chemical elements. Also, the horizontal coupling by the magnetic field, which reaches values of a few 10^5 G, is much more important than the effects of the horizontal turbulence. The field, however, is not sufficient to distort the shape of the equipotentials. We impose the condition that the energy of the magnetic field created by the Tayler–Spruit dynamo cannot be larger than the energy excess present in the differential rotation. This leads to a criterion for the existence of the magnetic field in stellar interiors.
Numerical tests are made in a rotating star model of 15 $M_\odot$ rotating with an initial velocity of 300 km s^{-1}. We find that the coefficients of diffusion for the transport of angular momentum by the magnetic field are several orders of magnitude larger than the transport coefficients for meridional circulation and shear mixing. The same applies to the diffusion coefficients for the chemical elements; however, very close to the core, the strong μ–gradient reduces the mixing by the magnetic instability to values not too different from the case without magnetic field. We also find that magnetic instability is present throughout the radiative envelope, with the exception of the very outer layers, where differential rotation is insufficient to build the field, a fact consistent with the lack of evidence of strong fields at the surface of massive stars. However, the equilibrium situation reached by a rotating star with magnetic field and rotation is still to be ascertained.

Key words. stars: rotation – stars: magnetic field – stars: evolution

1. Introduction

The inclusion of new physical effects in stellar evolution greatly improves the comparisons with observations. About twenty years ago the impact of mass loss by stellar winds on the evolution was found to be large. However, some significant discrepancies remained and the inclusion of rotation has enabled substantial progress in the comparisons with observed chemical abundances, with number counts and with chemical evolution of galaxies (cf. Langer et al. 1999; Maeder & Meynet 2000). The magnetic field is the next, but certainly not the last, in this series of effects which may influence all the outputs of stellar evolution.

In this work, we focus mainly on the relative importance of the effects of the magnetic field and of rotational instabilities to try to determine which effects can be let aside and what must be considered a priority. Section 2 summarizes the main effects of the magnetic field we are considering here following Spruit (1999, 2002). Section 3 compares the characteristic times of meridional circulation and of magnetic field instabilities. Section 4 considers what happens to the horizontal turbulence in presence of magnetic fields. Section 5 shows a new physical limit on the occurrence of the magnetic field in rotating stars. Section 6 gives some numerical values on the size of magnetic and rotational effects in the case of a 15 $M_\odot$ star. Section 7 presents the conclusions.

2. Basic properties of the magnetic field

Let us collect here some basic expressions and concepts we need below. Spruit (1999, 2002) has shown that the magnetic field can be created in radiative layers of stars in differential rotation. Even a small toroidal field is subject to an instability (called Tayler instability by Spruit), which creates a vertical field component. Differential rotation winds up this vertical component, so that many new horizontal field lines are produced. These horizontal field lines become progressively closer and denser in a star in a state of differential rotation, and therefore a much stronger horizontal field is built. This is the dynamo processs described by Spruit. The Tayler instability is a

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pinch–type instability. As shown by Spruit (1999), it has a very low threshold and is characterized by a short timescale. Also it is the first instability to occur. The magnetic shear instability may be present, but it is of much less importance (Spruit 1999).

The instability occurs in radiative zones and two cases are considered by Spruit (1999, 2002) depending on the thermal and μ–gradients through the associated oscillation frequencies,

$$N_{\rm T}^2 = \frac{g\delta}{H_{\rm P}}(\nabla - \nabla_{\rm ad}) \tag{1}$$

and

$$N_\mu^2 = \frac{g\varphi}{H_{\rm P}}\nabla_\mu. \tag{2}$$

The thermodynamic coefficients δ and φ are defined as follows $\delta = -\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_{\rm P,\mu}$ and $\varphi = \left(\frac{\partial \ln\rho}{\partial \ln\mu}\right)_{\rm T,P}$. The quantity $H_{\rm P}$ is the local pressure scale height. Following Spruit, we call case 0 the case where the μ–gradient dominates over the thermal gradient, i.e. when $N_\mu > N_{\rm T}$. Case 1 applies when the thermal gradient is the main restoring force, i.e. when $N_\mu < N_{\rm T}$. Let us point out that cases 0 and 1 are the two simpler limits of a more general case, where a proper account of both thermal and μ effects would be made. We may wonder whether we miss some important physical situations with this simplification. Indeed, numerical tests show that in practice the intermediate situation between cases 0 and 1 introduced by Spruit concerns a small but significant part of the star as shown in Fig. 2. In this part, cases 0 and 1 give diffusion coefficients which have the same order of magnitude, therefore the general effect is correctly described. However, there is little doubt that in future the general non-adiabatic case has to be examined in detail.

The growth rate of the magnetic instability is

$$\sigma = \frac{\omega_{\rm A}^2}{\Omega}, \tag{3}$$

where $\omega_{\rm A}$ is the Alfveń frequency, i.e. the frequency of magnetic waves. As shown by Pitts & Tayler (1986), the reduction factor $\frac{\omega_{\rm A}}{\Omega}$ in Eq. (3) results from the Coriolis force in a star rotating with angular velocity Ω. The Alfvén frequency is

$$\omega_{\rm A} = \frac{B}{(4\pi\rho)^{\frac{1}{2}}\ r}. \tag{4}$$

The magnetic instability works only if the unstable displacements do not lose too much energy against the stable stratification. For this to be the case, the radial displacements (against the buoyancy force) must be small compared to the horizontal displacements. Taking r as a maximum for the horizontal displacements, this sets an upper limit on the radial length scale $l_{r0,1}$ of the displacements,

$$l_{r0} < \frac{r\ \omega_{\rm A}}{N_\mu}, \tag{5}$$

and

$$l_{r1} < r\left(\frac{\Omega}{N_{\rm T}}\right)^{\frac{1}{2}}\left(\frac{K}{r^2\ \Omega}\right)^{\frac{1}{4}}, \tag{6}$$

where the indices 0 or 1 refer to the two cases considered. The term K is the thermal diffusivity $K = \frac{4acT^3}{3\kappa\rho^2 C_{\rm p}}$, where the other quantities have their usual meaning in stellar structure. There is also a minimal extent of the magnetic instability, below which it is quickly dissipated by magnetic diffusivity,

$$l_{r0,1}^2 > \frac{\eta\ \Omega}{\omega_{\rm A}^2}, \tag{7}$$

where η is the diffusivity of the magnetic field. The combination of Eq. (7) with Eqs. (5) and (6) leads to a minimum value of the Alfvén frequency for the occurrence of the magnetic instability in the two cases considered,

$$\left(\frac{\omega_{\rm A}}{\Omega}\right)_0 > \left(\frac{N_\mu}{\Omega}\right)^{\frac{1}{2}}\left(\frac{\eta}{r^2\ \Omega}\right)^{\frac{1}{4}}, \tag{8}$$

$$\left(\frac{\omega_{\rm A}}{\Omega}\right)_1 > \left(\frac{N_{\rm T}}{\Omega}\right)^{\frac{1}{2}}\left(\frac{K}{r^2\ \Omega}\right)^{\frac{1}{4}}\left(\frac{\eta}{K}\right)^{\frac{1}{2}}. \tag{9}$$

These are the conditions in order that the magnetic field overcomes the restoring force of buoyancy. In addition in the second case, the effects of the thermal diffusivity described by K are also accounted for. The corresponding maximum radial dimensions of the instabilities are given by the above Eqs. (5) and (6).

Spruit (2002; Eqs. (18) and (19)) considers the amplification timescale necessary to double the component B_φ starting from the radial component $B_{\rm r}$ over the largest characteristic lengths defined by Eqs. (5) and (6). He assumes the equality of the amplification timescale with the timescale for the damping by magnetic diffusivity over the above lenghtscales. In this way, he obtains the expressions for the Alfvén frequency. In the first case, where N_μ dominates, this is

$$\left(\frac{\omega_{\rm A}}{\Omega}\right)_0 = q\ \frac{\Omega}{N_\mu}, \tag{10}$$

where $q = -\frac{\partial \ln\Omega}{\partial \ln r}$. Thus, we see that the Alfvén frequency (a measure of the field amplitude) depends on the differential rotation parameter and on the ratio of the angular velocity to the Brunt–Väisälä frequency. When thermal diffusion is accounted for (N_μ negligible), one has

$$\left(\frac{\omega_{\rm A}}{\Omega}\right)_1 = q^{\frac{1}{2}}\left(\frac{\Omega}{N_{\rm T}}\right)^{\frac{1}{8}}\left(\frac{K}{r^2\ N_{\rm T}}\right)^{\frac{1}{8}}, \tag{11}$$

there the thermal diffusivity also intervenes. A few other recalls will be made when necessary.

3. Growth rate of the magnetic instability with respect to meridional circulation

A major question arises concerning a rotating star with a magnetic field. What happens to the meridional circulation in presence of the magnetic field of the Tayler–Spruit dynamo? Basically, meridional circulation occurs because thermal equilibrium cannot be achieved on an equipotential inside a rotating star. Thus, we may wonder whether the horizontal breakdown of thermal equilibrium in a rotating star can be compensated by a magnetic stress on an equipotential.

Let us define a velocity $U_{\rm magn}$ characterizing the radial growth of the magnetic instability. In the two cases 0 and 1, we

Table 1. Structural parameters of the model of 15 $M_\odot$ with $\upsilon_{\rm ini}$ = 300 km s^{-1} when $X_{\rm c}$ = 0.571.

$M_r/M_\odot$	$r/R_\odot$	Ω	$\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r}$	∇_μ	$\nabla_{\rm rad}$	$\nabla_{\rm ad}$	g	δ	$H_{\rm p}$	K
6.5726	1.445	0.76E-04	–0.11E+00	0.18E+00	0.317	0.337	0.86E+05	1.334	0.33E+11	0.35E+10
11.0386	2.080	0.69E-04	–0.33E+00	0.32E-03	0.269	0.348	0.69E+05	1.255	0.27E+11	0.79E+10
14.9421	4.159	0.60E-04	–0.84E-01	0.14E-06	0.225	0.326	0.23E+05	1.455	0.19E+11	0.81E+12

Table 2. Diffusion coefficients of the model of 15 $M_\odot$ with $\upsilon_{\rm ini}$ = 300 km s^{-1} when $X_{\rm c}$ = 0.571.

$M_r/M_\odot$	N_μ^2	$N_{\rm T}^2$	$D_{\rm ang0}$	$D_{\rm chem0}$	$D_{\rm ang1}$	$D_{\rm chem1}$	$D_{\rm ang1P}$	$D_{\rm chem1P}$	$D_{\rm shear}$	$U_{\rm circ}$
6.5726	0.48E-06	0.73E-07	0.12E+13	0.17E+09	0.15E+14	0.68E+10	0.55E+14	0.49E+11	0.40E+04	–0.19E-03
$U_{\rm magn}$				–0.66E+01		–0.77E+02		–0.29E+03		
11.0386	0.84E-09	0.26E-06	0.51E+19	0.31E+19	0.14E+14	0.15E+11	0.52E+14	0.10E+12	0.28E+07	–0.23E-04
$U_{\rm magn}$				–0.57E+08		–0.16E+03		–0.59E+03		
14.9421	0.17E-12	0.17E-06	0.16E+26	0.24E+28	0.29E+15	0.19E+12	0.15E+14	0.23E+10	0.19E+09	0.13E-03
$U_{\rm magn}$				–0.23E+14			–0.42E+03	–0.23E+02		

consider the ratio of the appropriate maximum lengths given by Eqs. (5) and (6) to the characteristic time σ^{-1},

$$U_{\rm magn,\,0,1} = \ l_{r\;0,1}\ \sigma. \tag{12}$$

For the case 0 with $N_\mu > N_{\rm T}$, one gets

$$U_{\rm magn,\,0} = \frac{r\ \omega_{\rm A}^3}{N_\mu\ \Omega}. \tag{13}$$

Then, using the above expression (Eq. (10)) for the Alfvén frequency, we obtain

$$U_{\rm magn,\,0} = q^3\ r\ \Omega\left(\frac{\Omega}{N_\mu}\right)^4. \tag{14}$$

The magnetic instability grows very fastly with the angular velocity and the differential rotation parameter q. In case 1 with $N_\mu < N_{\rm T}$, we get from Eq. (12) with Eqs. (6) and (3)

$$U_{\rm magn,\,1} = \ r\ \left(\frac{\Omega}{N_{\rm T}}\right)^{\frac{1}{2}}\left(\frac{K}{r^2\ \Omega}\right)^{\frac{1}{4}}\ \frac{\omega_{\rm A}^2}{\Omega}. \tag{15}$$

Using Eq. (11) for the Alfvén frequency, one has for the growth velocity of the magnetic instability

$$U_{\rm magn,\,1} = q\ r\ \Omega\left(\frac{\Omega}{N_{\rm T}}\right)^{\frac{1}{2}}\left(\frac{K}{r^2\ N_{\rm T}}\right)^{\frac{1}{2}}. \tag{16}$$

This velocity also grows with Ω and q, but less than in the case 0, because the radiative diffusivity reduces the dependence of the Alfvén frequency with respect to rotation parameters Ω and q.

These velocities have to be compared with the radial component of the radial part of the velocity of the meridional circulation $U(r)$, as given by Maeder & Zahn (1998; Eq. (4.38)). If the circulation velocity would largely dominates, this would mean that the magnetic field has effects which are too weak to influence the meridional circulation and the circulation would develop as usually supposed. If on the contrary, one has $U_{\rm magn,\,0,1} \gg U(r)$, this means that the magnetic instability develops much faster than the thermal instability at the origin of meridional circulation. If so, this means that the thermal instability created by rotation on an equipotential will interact firstly with the magnetic field. Tayler instability, which has the shortest timescale, may possibly develop and create a magnetic field which will introduce some stress horizontally on the equipotential. Detailed calculations must be done with accounting for the effects of the magnetic field in the equation for the entropy conservation at the basis for the calculations of meridional circulation, (this has been made for the effects of horizontal turbulence on the meridional circulation by Maeder & Zahn 1998).

For now, we assume in this case, in the whole region where $U_{\rm magn,\,0,1} \gg U(r)$, that the usual circulation velocity must be set to zero. This working hypothesis is largely confirmed by the comparison below (cf. Tables 1 and 2) of the velocities $U(r)$ of meridional circulation to the velocities $U_{\rm magn,0,1}$ characterizing the growth of the magnetic instability, the last ones being 4 to 7 orders of a magnitude larger than the first one.

Before looking more to the numerical values, we note that some caution has to be taken so that the comparisons of $U_{\rm magn,\,0,1}$ and $U(r)$ are done in a consistent way. The expression for the transfer of angular momentum by circulation and diffusion D is (see Zahn 1992; Maeder & Zahn 1998)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho r^2\overline{\Omega}\right)_{M_r} = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\rho r^4\overline{\Omega}U(r)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho\, D\, r^4\frac{\partial\overline{\Omega}}{\partial r}\right), \quad (17)$$

where $\overline{\Omega}(r)$ is the average angular velocity on the equipotential. From the second member of this expression, we see that to any diffusion coefficient D, one can define an associated velocity U_{D}

$$U_{\mathrm{D}} = 5;\frac{D}{\Omega}\;\frac{\partial\Omega}{\partial r} = 5\frac{D}{r}\;q. \quad (18)$$

The factor 5 results from the integration of the angular momentum over the star, which leads to the above Eq. (17). In order to be correct, the comparison of velocities must be made over the corresponding quantities as they appear in Eq. (17) for the angular momentum transfer, which in turn implies Eq. (18). Now, we take the coefficients of magnetic diffusion η_0 and η_1, which are obtained by replacing the inequalities of Eqs. (8) and (9) by equalities. This means (cf. Spruit 2002) that we consider the cases of marginal stability. Now, we can associate integrated velocities to these diffusion coefficients thanks to Eq. (18) and we find that they are just equal to the velocities $U_{\mathrm{magn,\,0}}$ and $U_{\mathrm{magn,\,1}}$ given above multiplied by a factor of 5. These velocities characteristic of the magnetic instability given by Eq. (18) are those to be compared to the velocity of meridional circulation, as it appears in the equation for the transport of angular momentum. As we shall see below, even if we omit these various considerations about Eq. (17), which result in the above factor of 5, the conclusions on the order of magnitude will be the same.

Below in Sect. 6, a star model with an initial mass of 15 $M_{\odot}$, a composition given by $X = 0.705$ and $Z = 0.02$ and an initial rotation velocity of 300 km s^{-1} has been computed. The prescriptions for rotation are those in Maeder & Meynet (2001). Tables 1 and 2 show the main parameters when the central H–content is $X_{\mathrm{c}} = 0.60$ at an age of 4.17×10^6 yr. We see that the velocities 5 $U_{\mathrm{magn,\,0,1}}$ are 3×10^4 to 3×10^7 times larger that the velocity $U(r)$ of meridional circulation. This shows that, even if present, the velocity of meridional circulation is negligible with respect to the corresponding velocity characterizing the transport of angular momentum by the magnetic field. Thus the question whether or not meridional circulation appears in the presence of magnetic field is of little relevance, since meridional circulation would anyway be totally negligible in comparison of the effect of magnetic instability. The same remark arises when one compares the magnetic diffusion coefficients for the angular momentum as given in Eqs. (40), (42) and (44) with the corresponding expression (28) for meridional circulation.

Thus, we suggest that in the stellar regions where magnetic field is present, *and only there* (see Sect. 5), the meridional circulation may be neglected with respect to magnetic field effects for the transport of angular momentum. As is shown in Sect. 4 below, the above conclusion also applies to the transport of the chemical elements.

4. The importance of magnetic stress vs. horizontal turbulence

4.1. Horizontal coupling

The turbulence in rotating stars is highly anisotropic and has a strong horizontal component, described by a diffusion coefficient D_{h}, because vertically the thermal gradient stabilizes turbulence. This horizontal turbulence strongly reduces the horizontal differential rotation, so that rotation varies only radially. The rotation is therefore said to be “shellular” (Zahn 1992, Ω uniform at the surface of isobaric shells). The coefficient D_{h} also plays a role in the mixing of the chemical elements and meridional circulation (Maeder & Zahn 1998). A first expression for D_{h} was given by Zahn (1992). Recently another expression of this coefficient has been obtained (Maeder 2003),

$$D_{\mathrm{h}} = A\; r\;\left[r\overline{\Omega}(r)\;V\;\left(2V-\alpha U\right)\right]^{\frac{1}{3}}, \quad (19)$$

with $A \approx 0.10$, $V(r)$ the horizontal component of meridional circulation and $\alpha = \frac{1}{2}\frac{\mathrm{d}\ln r^2\Omega}{\mathrm{d}\ln r}$. Typically, this new estimate of D_{h} is of the order of 10^{11} to 10^{12} cm^2 s^{-1} in a massive star, i.e. typically 10^2 to 10^3 times larger than the coefficient previously estimated. Recent developments by Zahn (private communication) and entered into model calculations by Palacios (private communication) confirm this higher values of D_{h}.

The question is now: what happens to this horizontal turbulence, even if it is much larger than initially supposed, in the presence of the magnetic field? According to Spruit (2002), the Tayler instability and the associated dynamo leads to horizontal field components in cases 0 and 1,

$$B_{\varphi,0} = (4\,\pi\,\rho)^{\frac{1}{2}} r\; q\;\frac{\Omega^2}{N_\mu}, \quad (20)$$

$$B_{\varphi,1} = (4\,\pi\,\rho)^{\frac{1}{2}} r\;\Omega\; q^{\frac{1}{2}}\left(\frac{\Omega}{N_{\mathrm{T}}}\right)^{\frac{1}{8}}\left(\frac{K}{r^2N_{\mathrm{T}}}\right)^{\frac{1}{8}}. \quad (21)$$

For the numerical model of Tables 1 and 2, we find an horizontal field

$$B_{\varphi,0} = 4.35\times10^5\ \mathrm{G} \quad \text{at } M_r/M_{\odot} = 6.57, \quad (22)$$

$$B_{\varphi,1} = 2.80\times10^5\ \mathrm{G} \quad \text{at } M_r/M_{\odot} = 11.04. \quad (23)$$

These fields are high with respect to our current standards, but the magnetic pressure,

$$P_{\mathrm{magn}} = \frac{B^2}{8\,\pi} \quad (24)$$

is small with respect to the total pressure. One has typically a ratio $P_{\mathrm{magn}}/P = 1.5\times10^{-6}$, 2.7×10^{-6} in the above examples. Thus, the magnetic field generated by the Tayler–Spruit mechanism does not affect the shape of the equipotentials. The ratios $B_{\mathrm{r}}/B_{\varphi}$ of the radial to the horizontal field components are given by Eqs. (21) and (23) by Spruit (2002) and the values are of the order of 10^{-3} to a few 10^{-2}.

Now, the question is how does the horizontal coupling due to the magnetic field $B_{\varphi,0,1}$ compares with the horizontal turbulence characterized by D_{h}. Spruit (2002; Eqs. (31) and (32)) has given an estimate of the coefficient of the vertical coupling by magnetic field. However, we cannot use it here, because the field is very anisotropic. The coefficient $D_{\mathrm{B_h}}$ for the horizontal coupling must be much larger than the vertical one. At low rotation, we suggest to take the following estimate,

$$D_{\mathrm{B_h}} \approx r^2 \, \omega_{\mathrm{A}} \approx \frac{r \, B_\varphi}{(4\pi \, \rho)^{\frac{1}{2}}}. \tag{25}$$

At high rotation, the growth rate is also reduced by the Coriolis factor $\omega_{\mathrm{A}}/\Omega$ (as suggested by Spruit in a private communication), thus one has then

$$D_{\mathrm{B_h}} \approx r^2 \, (\omega_{\mathrm{A}}^2/\Omega) \approx \frac{B_\varphi^2}{4\pi \, \rho \, \Omega}. \tag{26}$$

Interestingly enough, this coefficient is of the order of 10^{14} cm^2 s^{-1} in the above examples. Thus, the horizontal magnetic coupling is about 10^3 larger than the horizontal coupling by turbulence, even when we take the value given by the new coefficient D_{h} in Eq. (19) above. Therefore, our conclusion is that in the presence of Tayler–Spruit dynamo, we can anyway neglect the coupling due to horizontal turbulence with respect to the coupling insured by the horizontal component of the magnetic field. More likely, we suspect that the horizontal turbulence may not develop in this case or occur in a very different and limited way.

4.2. Remarks on further consequences: The case of D_{eff}

In models with rotation and without magnetic fields, the combined effect of meridional circulation and horizontal turbulence may be treated as a diffusion with a coefficient usually called D_{eff} (cf. Chaboyer & Zahn 1992),

$$D_{\mathrm{eff}} = \frac{[r \, U(r)]^2}{30 \, D_{\mathrm{h}}}. \tag{27}$$

Now, since D_{h} is negligible, we may wonder what happens to the transport of chemical elements by meridional circulation. As well known, it is in general not correct to express an advection as a diffusion, but let us do it here just for the purpose of comparing orders of magnitude. The diffusion coefficient we may associate to the meridional circulation is in this case,

$$D_{\mathrm{circ}} \approx |r \, U(r)|. \tag{28}$$

This must be compared to the coefficients $D_{\mathrm{chem},\,0,\,1}$, which are obtained by imposing equality in Eqs. (8) and (9) and taking $D_{\mathrm{chem},\,0,\,1} = \eta_{0,\,1}$. The equality means that one considers the case of marginal stability as determining for the transport. Then, in the obtained relations, the Alfvén frequency is expressed with the help of Eqs. (10) and (11). In this way, Spruit obtains (2002; Eqs. (42) and (43)),

$$D_{\mathrm{chem},\,0} = r^2 \, \Omega \, q^4 \left(\frac{\Omega}{N_\mu}\right)^6 , \tag{29}$$

and

$$D_{\mathrm{chem},\,1} = r^2 \, \Omega \, q \left(\frac{\Omega}{N_{\mathrm{T}}}\right)^{\frac{3}{4}} \left(\frac{K}{r^2 \, N_{\mathrm{T}}}\right)^{\frac{3}{4}}. \tag{30}$$

In the numerical example of Tables 1 and 2, we find ratios $\frac{D_{\mathrm{circ}}}{D_{\mathrm{chem}}} = 0.11, 3.2 \times 10^{-5}, 1.9 \times 10^{-4}$ at the 3 layers considered, (taking in each case the appropriate expressions for the case 0 and 1). Thus, we see that through most of the star the circulation (if it exists!) would have a negligible effect for the transport of chemical elements. Only close to the core, the circulation might represent about 10% of the transport by the magnetic instability. This is small anyway and owing to the profound doubts expressed above on the occurrence of any circulation, we consider that any transport of chemical elements by meridional circulation can generally be neglected in the presence of Tayler–Spruit dynamo.

5. An energy condition for the magnetic field

5.1. Energy condition

In a radiative zone, the magnetic field created by the Tayler–Spruit instability arises from differential rotation. Therefore, we must impose the condition that *the energy in the magnetic field cannot be larger than the excess energy in differential rotation*. Tayler (1973) and Pitts & Tayler (1986) have studied the conditions for the appearance of the magnetic instabilities, which have very short growth times. The above condition is different, it is an energy condition, which must be satisfied on longer timescales. Due to the magnetic diffusivity, the field once created tends to disappear. Taking the values of $D_{\mathrm{chem}} = \eta$ given for example in Fig. 6 (see also Tables 1 and 2), which are between $10^{10.5}$ and 10^{12} cm^2 s^{-1} over most of the star except very close to the convective core, we find that the timescale $\tau \approx \frac{R^2}{\eta}$ for the diffusion of the magnetic field is of the order of 2×10^3 to 8×10^4 yr, which is short with respect to the stellar lifetimes. This means that the field created by Tayler–Spruit dynamo will exist only if the energy of the magnetic field is continuously replenished by differential rotation during the MS evolution. Therefore, we need to apply the above condition.

For a magnetic instability with a displacement of amplitude ξ, the kinetic energy E_B by unit of mass is

$$E_B = \frac{1}{2} \, \omega_{\mathrm{A}}^2 \, \xi^2. \tag{31}$$

The excess energy E_Ω in the differential rotation is the difference of energy between the existing flow with differential rotation and a flow with an average rotation over the considered radial distance r. Let us consider a horizontal velocity field $W(r)$. Over a vertical distance dr, the energy excess dE_Ω over the extent of the magnetic instability is

$$\begin{aligned} \mathrm{d}E_\Omega &= \frac{1}{2}\left[W^2 + (W + \mathrm{d}W)^2\right] - \frac{1}{2} \cdot 2\left(W + \frac{\mathrm{d}W}{2}\right)^2 \\ &= \frac{\mathrm{d}W^2}{4}. \end{aligned} \tag{32}$$

If l_{r} and l_{h} are the radial and horizontal components of the displacement ξ of the magnetic oscillation (cf. Spruit 2002), the energy excess over the displacement ξ can be written,

$$E_\Omega = \frac{1}{4}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 \xi^2 \left(\frac{l_{\mathrm{r}}}{l_{\mathrm{h}}}\right)^2. \tag{33}$$

The horizontal velocity $W(r) = r \sin\vartheta\ \Omega$, where ϑ is the colatitude. Thus, one has

$$\frac{\mathrm{d}W}{\mathrm{d}r} = r \sin\vartheta\ \frac{\mathrm{d}\Omega}{\mathrm{d}r} = \Omega \sin\vartheta\ \frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r} = -\Omega \sin\vartheta\ q. \tag{34}$$

The condition $E_\Omega > E_B$ leads to

$$\frac{1}{2}\Omega^2 q^2 \sin^2\vartheta\ \left(\frac{l_\mathrm{r}}{l_\mathrm{h}}\right)^2 > \omega_\mathrm{A}^2. \tag{35}$$

This would apply to a given colatitude ϑ. This equation as it stands means that the reservoir of available rotational energy is larger at the equator, while we know (cf. Spruit 1999) on the other hand that Tayler instability is stronger away from equator. Now, for the ratio $(\frac{l_\mathrm{r}}{l_\mathrm{h}})$ of the vertical to the horizontal displacement, we could take for example in the outer layers, where differential rotation is generally small, the value given by Eq. (6) above (case 1). This promptly leads to the following criterion with account of Eq. (11),

$$|q| > 3\ \left(\frac{N_\mathrm{T}}{\Omega}\right)^{\frac{1}{4}} \left(\frac{r^2 N_\mathrm{T}}{K}\right)^{\frac{1}{4}}. \tag{36}$$

This equation ignores the geometry of the field. However, we have to consider carefully the geometry of the problem in 2 specific respects:

1. We must account for the fact that in the model of shellular rotation the energy of rotation is not a local quantity depending on (r, ϑ), but on r only. The physical reason in the usual rotating models is the strong horizontal turbulence (cf. Zahn 1992). In the present models, the horizontal magnetic coupling is even stronger, as seen above in Sect. 4.1, so that shellular rotation is a valid assumption here. In such a case, the average stellar structure of the rotating star corresponds very well to the structure at a colatitude ϑ given by the root of the second Legendre polynomial $P_2(\cos\vartheta) = 0$, (this has been verified in a recent work, Maeder 2001). Thus it is appropriate to consider for the differential rotation $\frac{\mathrm{d}W}{\mathrm{d}r}$ in Eq. (34) on a given equipotential the average value $\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 = \frac{2}{3}\ \Omega^2 q^2$.

2. The geometry of the field is also particular (cf. Spruit 1999, 2002). It consists of stacks of magnetic loops concentric with the rotation axis. The main component of the displacement due to the Tayler instability is perpendicular to the rotation axis. This means that at colatitude ϑ, the ratio $(\frac{l_\mathrm{r}}{l_\mathrm{h}})$ behaves essentially as $\tan\vartheta$. Since the polar caps are most unstable, while the equatorial regions are not, it is clear that in the polar regions one has a ratio $(\frac{l_\mathrm{r}}{l_\mathrm{h}})$ smaller than 1. However, in 1D models as here we must consider the significant average for the whole range of colatitudes. The field behaves as $B_\varphi \approx \sin\vartheta \cos\vartheta$ (cf. Spruit 1999; Eq. (35)) and the Tayler instability develops only for $\vartheta \leq \pi/4$, therefore it is necessary on a given isobar to consider colatitudes ϑ smaller or at most equal to $\pi/4$. If we take this last value as the limit, this gives the upper bound $\tan\vartheta = (\frac{l_\mathrm{r}}{l_\mathrm{h}}) \leq 1$.

From these two geometrical remarks, we obtain the necessary condition for the existence of a magnetic field generated by the Tayler–Spruit dynamo as follows,

$$\frac{\omega_\mathrm{A}}{\Omega} < \frac{|q|}{\sqrt{3}}. \tag{37}$$

This means that the degree of differential rotation q must at least be larger than $\sqrt{3}$ times the ratio $\frac{\omega_\mathrm{A}}{\Omega}$ of the Alfvén to the rotation frequency, in order that there is enough energy in the differential rotation to allow the Tayler–Spruit dynamo to operate and build a magnetic field. We insist that this is a necessary condition. If this condition is not realized, there is certainly no magnetic field created by the dynamo. Further work may perhaps lead to an even more constraining condition. The numerical factor, here $\sqrt{3}$, may depend on the exact geometry of the magnetic displacements in a rotating star. We note also that if there are several types of instabilities generated by differential rotation, the available energy given by Eq. (33) would in some way be shared between the instabilities. However, as mentioned above, the Tayler instability is the main one and shear instabilities appear negligible in comparison.

We can go a step further, since the Alfvén frequency ω_A is a function of rotation and differential parameter $|q|$. For the case 0, where the μ–gradient dominates, the ratio $\frac{\omega_\mathrm{A}}{\Omega}$ is given by the above Eq. (10). Thus, the above condition (37) becomes in this case

$$\frac{\Omega}{N_\mu} < \frac{1}{\sqrt{3}} = 0.5774. \tag{38}$$

This is the necessary condition in order that a magnetic field may develop from differential rotation in case 0. At first glance, this condition may look strange, since it means that Ω must be smaller that some value. The reason is that ω_A grows like $q\ \frac{\Omega^2}{N_\mu}$, while the upper limiting ω_A expressed by Eq. (37) goes like $\frac{|q|}{\sqrt{3}}\ \Omega$. Thus, if Ω would be too big, the actual ω_A would overcome the critical value. In the numerical examples of Tables 1 and 2, case 0 is relevant at the edge of the core at $M_\mathrm{r}/M_\odot = 6.57$. There $\frac{\Omega}{N_\mu}$ is 0.11, (typically $\frac{\Omega}{N_\mu}$ lies between 0.1 and 0.15 throughout the star). This is smaller than 0.5774 and thus the magnetic field can be present in these layers.

We now consider case 1 with thermal diffusion, the Alfvén frequency is given by Eq. (11). With the above condition (37), one obtains

$$\frac{|q|}{3} > \left(\frac{\Omega}{N_\mathrm{T}}\right)^{\frac{1}{4}} \left(\frac{K}{r^2\ N_\mathrm{T}}\right)^{\frac{1}{4}}. \tag{39}$$

This means that the differential rotation parameter $|q|$ has to be large enough to be able to generate the magnetic field. In the numerical examples in Sect. 6, the condition is not satisfied in the very outerlayers, which have a too weak differential rotation.

5.2. Collection of formulae and recipes

Let us collect here the various expressions we have for the diffusion coefficients by the Tayler–Spruit dynamo in radiative zones. Case 0 applies when $N_\mu > N_\mathrm{T}$. Magnetic field is present only when the criterion given by Eq. (38) is satisfied. Then the diffusion coefficients for the transport of the angular momentum and chemical elements are respectively,

$$D_\mathrm{ang0} = r^2 \Omega q^2 \left(\frac{\Omega}{N_\mu}\right)^4, \tag{40}$$

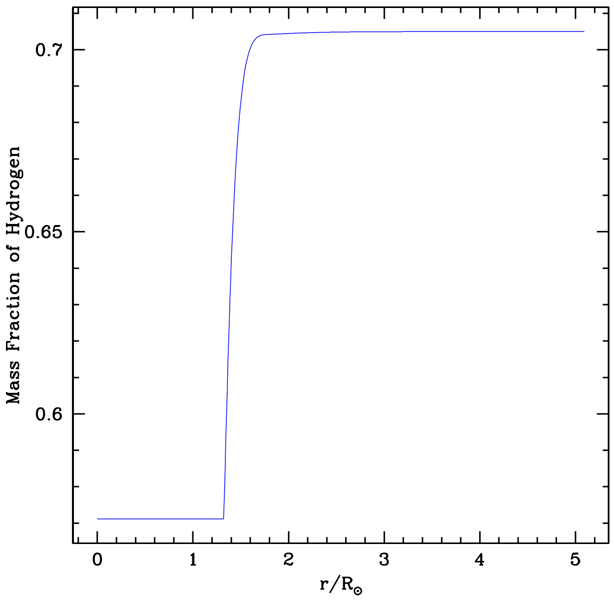

Fig. 1. Internal H–profile in the 15 $M_\odot$ test model with $v_{\rm ini}$ = 300 km s^{-1}. The model is at an age of 4.0×10^6 yr. This is the reference model in which we examine the properties of the magnetic field in detail.

$$D_{\rm chem0} = r^2 \Omega q^4 \left(\frac{\Omega}{N_\mu}\right)^6 . \tag{41}$$

Case 1 applies when $N_\mu < N_{\rm T}$, then account is given to the thermal diffusivity. Magnetic field is present only when the criterion given by Eq. (39) is satisfied. The diffusion coefficients for the angular momentum and chemical elements are respectively

$$D_{\rm ang1} = r^2 \Omega \left(\frac{\Omega}{N_{\rm T}}\right)^{1/2} \left(\frac{K}{r^2 N_{\rm T}}\right)^{1/2} , \tag{42}$$

$$D_{\rm chem1} = r^2 \Omega \; |q| \left(\frac{\Omega}{N_{\rm T}}\right)^{3/4} \left(\frac{K}{r^2 N_{\rm T}}\right)^{3/4} . \tag{43}$$

Normally, these coefficients are larger than those which would be obtained in case 0 with $N_{\rm T}$ instead of N_μ, because the account for thermal effects reduces the buoyancy force which opposes to the magnetic instability. However, as noted by Spruit (2002), it may happen in some cases that these coefficients with index "1" are smaller. As noted by Spruit (2002), this is an artefact from the simplification introduced by considering only the 2 limiting cases 0 and 1. Spruit suggests to introduce an interpolation formula depending on q. We hesitate to do so, because this extra–dependence on the differential rotation parameter q is unphysical, since the interpolation should rather depend on the thermal and magnetic diffusivities K and η. Thus, the suggested treatment might introduce spurious effects in some evolutionary stages. The general case where thermal effects and μ–gradient are accounted for needs to be worked out in future. For now, we prefer to do the following by considering the coefficients

$$D_{\rm ang1P} = r^2 \Omega q^2 \left(\frac{\Omega}{N_{\rm T}}\right)^4 , \tag{44}$$

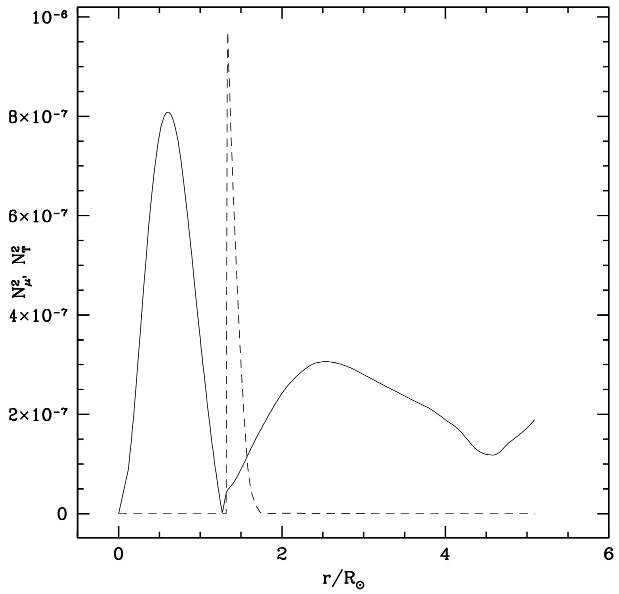

Fig. 2. The oscillation frequencies in the model of Fig. 1. N_μ^2 is indicated by a continuous line and $N_{\rm T}^2$ is given by a dashed line.

$$D_{\rm chem1P} = r^2 \Omega q^4 \left(\frac{\Omega}{N_{\rm T}}\right)^6 . \tag{45}$$

These are the same equations as in case 0, but with $N_{\rm T}$ instead of N_μ. This means that we are considering only the restoring force of the thermal gradient and that we are ignoring the non–adiabatic radiative losses.

For case 1, we compare the coefficient with indices "1" and "1P" and must take the larger ones. From Fig. (5) below, we see that the coefficients "1P" may be larger than the coefficients "1" in some parts of the star. Of course, we have also to test whether the magnetic field can be created from differential rotation. For such zones of case 1P, the criterion for the existence of the magnetic field is evidently the following one

$$\frac{\Omega}{N_{\rm T}} < \frac{1}{\sqrt{3}} = 0.5774, \tag{46}$$

and the appropriate test has to be made in the concerned layers.

6. Numerical applications to stellar models

6.1. Magnetic vs. rotational transports of angular momentum and of chemical elements

We calculate evolutionary models of a 15 $M_\odot$ star with initial composition $X = 0.705$ and $Z = 0.02$. The physics of the models (opacities, nuclear reactions, mass loss rates, treatment of rotation, increase of the mass loss rates with rotation, etc.) are the same as in Meynet & Maeder (2003). We compute the MS evolution of a model with an initial velocity of 300 km s^{-1}, which leads to an average velocity during the MS phase of about 220 km s^{-1}, which corresponds to the observed average rotation velocity. We now consider in detail the properties of a particular model with rotation to see the various coefficients and criteria characterizing the growth of the magnetic field. We take the model at an age 4.17×10^6 yr with a central H–content

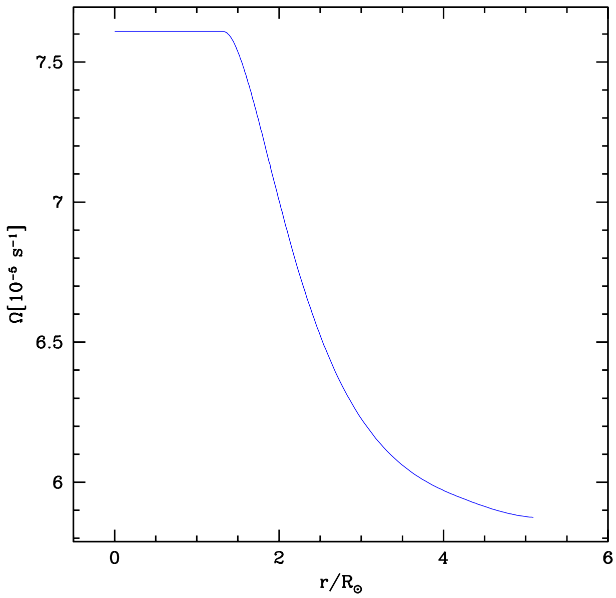

Fig. 3. The distribution of the angular velocity in the reference model of Fig. 1 with rotation (continuous line).

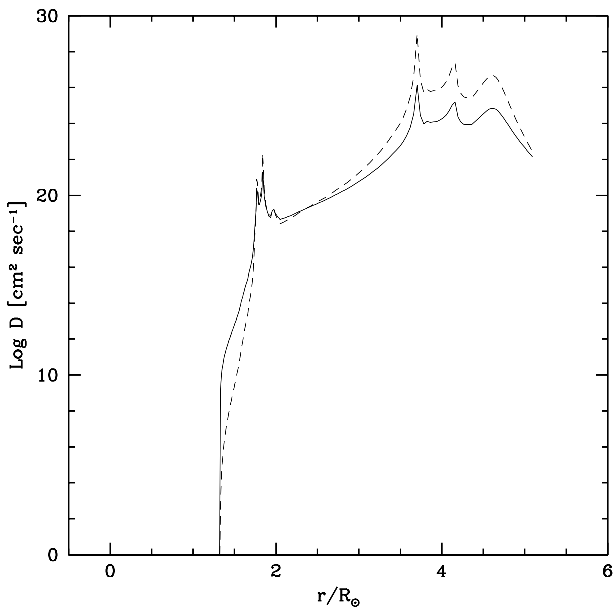

Fig. 4. The diffusion coefficients $D_{\rm ang0}$ (continuous line) and $D_{\rm chem0}$ (dashed line) corresponding to case 0. As discussed in the text, these coefficients apply only in the rising part between $r/R_\odot$ = 1.32 and 1.59.

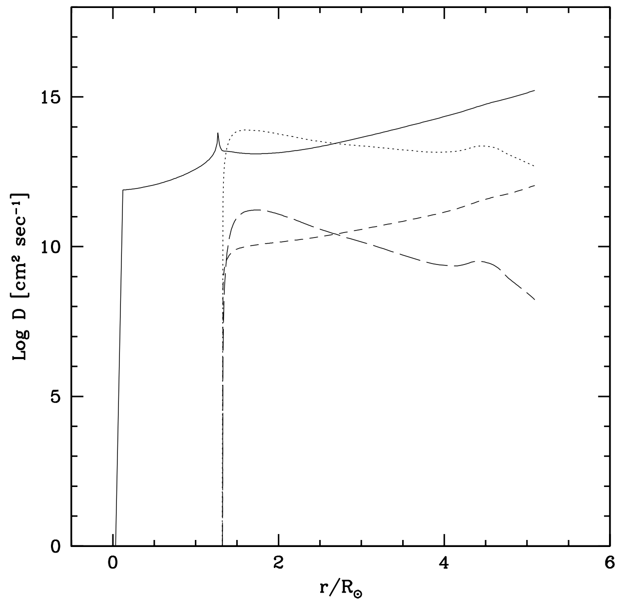

Fig. 5. The diffusion coefficients for angular momentum $D_{\rm ang1}$ (continuous line) and $D_{\rm ang1P}$ (dotted line). The diffusion coefficients for chemical elements $D_{\rm chem1}$ (dashed line) and $D_{\rm chem1P}$ (long dashed line). In each case, the largest diffusion coefficient has to be taken.

$X_{\rm c}$ = 0.571. The H–profile inside the star is illustrated in Fig. 1 and the oscillation frequencies N^2_μ and $N^2_{\rm T}$ in Fig. 2. We see that case 0 applies between $r/R_\odot$ = 1.32 and 1.59, which corresponds to mass coordinates 5.53 and 7.70 $M_\odot$ respectively. The internal profile $\Omega(r)$ is illustrated in Fig. 3.

The diffusion coefficients $D_{\rm ang0}$ and $D_{\rm chem0}$ are illustrated in Fig. 4. These coefficients grow very fastly at the edge of the core since N^2_μ decreases very fastly there and the dependence in the ratio $(\frac{\Omega}{N_\mu})$ goes like the power 4 and 6 for the two coefficients respectively. These coefficients apply only in the rising part between $r/R_\odot$ = 1.32 and 1.59 as indicated. Above this value, the main restoring force is no longer the μ–gradient, but the stable temperature gradient.

In case 1, where thermal gradients dominate, the diffusion coefficients are illustrated in Fig. 5. We see that for the transports of angular momentum and of chemical elements, the coefficients with indices "1" dominate in the external parts of the star above $r/R_\odot$ = 2.69. However, we also notice that in a sizeable region, i.e. between $r/R_\odot$ = 1.59 and 2.69, the coefficients with indices "1P" dominate over those with "1". This occurs, as expected, at some limited distance above the edge of the convective core, at the place where the μ–gradient becomes small enough, but is still different from zero. In these regions, more general developments having case 0 and 1 as limiting cases would be a progress. We see that the differences between the two cases "1P" and "1" amounts to a maximum of 0.4 dex and 0.6 dex for the transport of angular momentum and chemical elements respectively. This is limited, but non negligible, and it may justify a further study of the physics of the general case. However, we notice that this difference is small when compared to the differences resulting from the inclusion of the magnetic field or not, which as shown below amounts to several orders of magnitude. Thus, we conclude that the present coefficients of diffusion need to be further improved, but they nevertheless describe correctly the main results of the inclusion of the magnetic field.

Tables 1 and 2 provides some useful structural parameters, the diffusion coefficients and the velocities of meridional circulation and of magnetic instability at 3 locations in the reference model of 15 $M_\odot$ with an initial velocity of 300 km s^{-1} at an age 4.17×10^6 yr. The three levels considered illustrate the case 0, 1P and 1 respectively. These Tables permit further quantitative analysis of the various terms intervening in the equations.

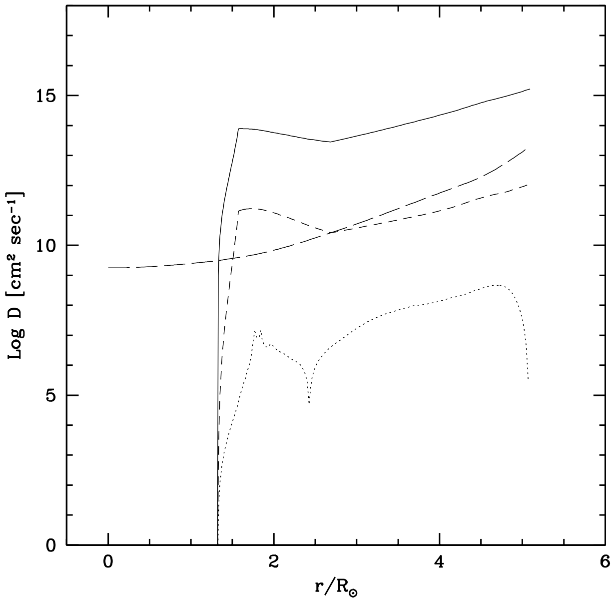

Fig. 6. The figure shows the complete description of the diffusion coefficient D_{ang} for angular momentum (continuous line) made according to prescriptions of Sect. 5.2. The complete description of the diffusion coefficient D_{chem} for chemical elements is given by the dashed line. The dotted line shows the shear diffusion coefficient D_{shear} from Maeder (1997). The thermal diffusivity coefficient K is represented by a long–dashed line.

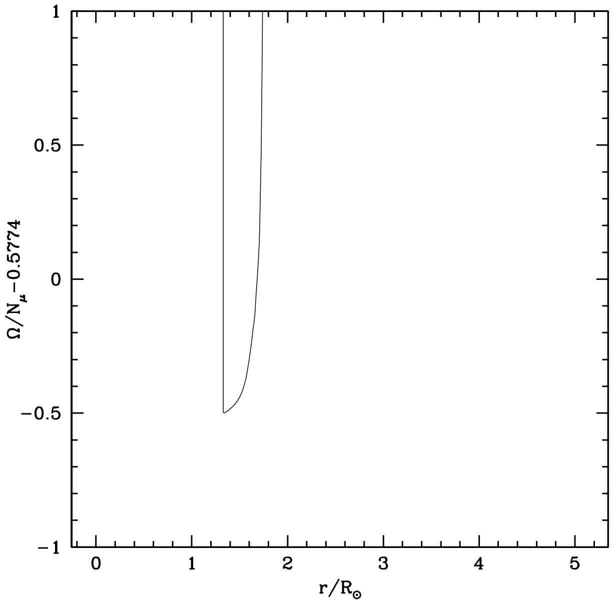

Fig. 7. The difference $\frac{\Omega}{N_\mu} - \frac{1}{\sqrt{3}}$ when case 0 applies, i.e. in the region between $r/R_\odot = 1.32$ and 1.59. According to criterion (38), if this difference is negative, magnetic field is created. Thus, we see that in the region where the μ–gradient dominates, magnetic field can be generated by the Tayler–Spruit dynamo.

Figure 6 shows the comparison of the diffusion coefficients due to the magnetic field compared to the diffusion coefficient D_{shear} by shear instability in the rotating star and to the thermal diffusivity K. The transport of angular momentum by the magnetic field is 6−7 orders of magnitude stronger than by shear instability in a rotating star. Similarly, as discussed in Sect. 4.2, the magnetic transport of angular momentum is also 4−7 orders of magnitude larger than by meridional circulation. Therefore, we conclude that transport of angular momentum by magnetic field is totally dominating, if magnetic field is present.

For the transport of chemical elements, the difference between the two diffusion coefficients amount to 3−5 orders of magnitude in favour of the transport by magnetic field. The difference is especially large in deep regions at some distance of the convective core. At the very edge of the convective core, the dependence of the coefficient $D_{\text{chem},0}$ in the power 6 of $(\frac{\Omega}{N_\mu})$ reduces the magnetic diffusion drastically, so that as mentioned in Sect. 4.2 the ratio of the transport of chemical elements by the magnetic instability to the transport by circulation may amount to about 1 order of magnitude. Some tests indicate that this makes the chemical enrichments in helium and nitrogen at the stellar surface are stronger, but not too different, from those without magnetic field, despite the fact that the diffusion coefficients with magnetic field are orders of magnitude larger over most of the stellar interior. On the whole, we see that for chemical mixing also, the magnetic instability plays a great role.

It is also interesting to see that diffusion coefficients by the magnetic field are almost equal (transport of chemical elements) or even larger (transport of the angular momentum) than the thermal diffusivity. This means that magnetic effects by the Tayler–Spruit dynamo are in general equal or larger than thermal effects. Globally thermal effects may be relatively more significant in the outer layers.

6.2. Existence of a magnetic field due to the energy condition of differential rotation

An important question in the models is to determine at each shell mass whether differential rotation described by parameter q is sufficient to create the Tayler–Spruit dynamo to produce a magnetic field. We examine here whether these conditions are fulfilled in the model with rotation only studied in the previous subsection. Firstly, we examine the regions adjacent to the core between $r/R_\odot = 1.32$ and 1.59, where case 0 applies, since N_μ dominates. Figure 7 shows the difference $\frac{\Omega}{N_\mu} - \frac{1}{\sqrt{3}}$, we see that in the concerned region between $r/R_\odot = 1.32$ and 1.59, this difference is negative, thus magnetic field is present there. This is an interesting result because it means that despite the strong restoring buoyancy force due to the very large μ–gradient, the differential rotation is high enough to develop magnetic instability. At each time step during evolution, such tests need to be performed.

Secondly, we examine the zone above $r/R_\odot = 1.59$ and up to $r/R_\odot = 2.69$, where case 1P applies. There, we check for the difference $\frac{\Omega}{N_{\text{T}}} - \frac{1}{\sqrt{3}}$. If this expression is negative, magnetic field is created. This expression lies between −0.40 and −0.50 in this whole intermediate region. Thus, we conclude that magnetic field is also present there.

The criterion for the existence of the magnetic field in the external zone, which corresponds to case 1 is given by Eq. (39). From Fig. 8, we see that in this zone which lies between

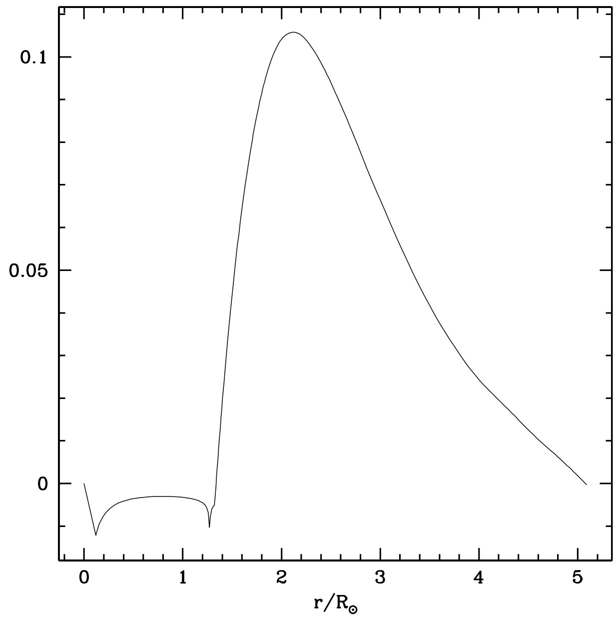

Fig. 8. The quantity $\frac{|q|}{3} - \left(\frac{\Omega}{N_T}\right)^{\frac{1}{4}} \left(\frac{K}{r^2\, N_T}\right)^{\frac{1}{4}}$ given by Eq. (39) vs. $r/R_\odot$. When this quantity is positive in the region where case 1 applies, i.e. above $r/R_\odot = 2.69$, differential rotation is sufficient to generate the magnetic field, which is the case here.

$r/R_\odot = 2.70$ and the surface, the difference $\frac{|q|}{3} > \left(\frac{\Omega}{N_T}\right)^{\frac{1}{4}} \left(\frac{K}{r^2\, N_T}\right)^{\frac{1}{4}}$ is generally positive, which means that magnetic field is present over that region. However, we notice that very close to the surface this difference goes to zero. This means that at the surface, differential rotation is becoming insufficient to generate the magnetic instability. This is interesting because it may explain why there is in general no strong magnetic field observed at the surface of OB stars (cf. Mathys 2003), despite the likely existence of a strong internal field created the Tayler–Spruit instability.

These results show that in a rotating star, the conditions on the differential rotation for the growth of the magnetic field are largely realized, with the possible exception of the superficial layers.

7. Conclusions

The main conclusion is that the Tayler instability and the Tayler–Spruit dynamo are of major importance for stellar evolution, both for the transport of angular momentum and for the transport of chemical elements. Future evolutionary models applying the results of this work will be made to study the results on tracks, surface composition, rotation, etc.

It is likely that in a rotating star calculated with magnetic field from the beginning, the differential rotation is very much reduced by the magnetic transport of angular momentum described above. We may suspect that differential rotation will be reduced down to a stage where the criterion (37) discussed in Sect. 5.1 is just marginally satisfied, i.e.

$$|q| = \sqrt{3}\ \frac{\omega_A}{\Omega}. \tag{47}$$

Indeed, if differential rotation is higher than given by this criterion, magnetic field develops and the associated coupling reduces differential rotation. If, at the opposite, differential rotation is lower than given by criterion (47), the growth of mean molecular weight by nuclear reactions in the central regions together with angular momentum conservation will produce an enhancement of differential rotation. Thus, a stage of marginal equilibrium is most likely reached during MS evolution. It may also be that the outer layers never have sufficient differential rotation to build Tayler-Spruit dynamo. Further numerical models will explore the evolution of stars with rotation and magnetic field, and analyse the coupling between magnetic field and differential rotation.

Note added in proof: The above calculations determine the magnetic field which develops in a rotating star, which had no field until a considered specific evolutionary stage. Recent calculations have confirmed, as suggested in the conclusions, that models with magnetic field included throughout MS evolution reach an equilibrium situation with very little differential rotation. This happens in turn to make a feedback on the field amplitude and on the velocity of meridional circulation. This has two consequences: a) the internal magnetic fields are reduced to a few 10^4 G; b) the velocity of meridional circulation is increased to about 10^2 cm s^{-1}. This confirms that magnetic fields play an essential role. However, in the equilibrium models some effects of meridional circulation could also influence the internal profile of $\Omega(r)$ due to the above feedback.

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