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Rossby waves in "shallow water" magnetohydrodynamics

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ABSTRACT

Aims. The influence of a toroidal magnetic field on the dynamics of Rossby waves in a thin layer of ideal conductive fluid on a rotating sphere is studied in the "shallow water" magnetohydrodynamic approximation for the first time.
Methods. Dispersion relations for magnetic Rossby waves are derived analytically in Cartesian and spherical coordinates.
Results. It is shown that the magnetic field causes the splitting of low order (long wavelength) Rossby waves into two different modes, here denoted fast and slow *magnetic Rossby waves*. The high frequency mode (the fast magnetic Rossby mode) corresponds to an ordinary hydrodynamic Rossby wave slightly modified by the magnetic field, while the low frequency mode (the slow magnetic Rossby mode) has new and interesting properties since its frequency is significantly smaller than that of the same harmonics of pure Rossby and Alfvén waves.

Key words. magnetohydrodynamics (MHD) – waves

1. Introduction

The large-scale dynamics of planetary atmospheres is mostly determined by Rossby waves. These waves arise because of the latitudinal variation of the Coriolis parameter and are widely used in the geophysical context (Pedlosky 1987; Gill 1982). Rossby waves also can be of importance in solar and stellar astrophysics, particularly in a thin layer called *tachocline* that is believed to exist below the convection zone of solar-like stars (Spiegel & Zahn 1992; Gough & McIntyre 1998; Garaud 2002; Cally 2003; Miesch 2005). The thickness of the tachocline is very small compared to the stellar radius and, therefore, the ordinary shallow water approximation can be easily applied, but the hydrodynamic (HD) Rossby wave theory needs to be modified in the presence of a large-scale horizontal magnetic field. The influence of the horizontal magnetic field on the large-scale fluid dynamics has been studied in the context of the Earth's liquid core using the two dimensional β-plane approximation in Cartesian coordinates by Hide (1966). However, to study the plasma dynamics over spatial scales comparable to the stellar radius requires to consider spherical coordinates. Magnetohydrodynamic (MHD) "shallow water" equations for the solar tachocline have been recently proposed by Gilman (2000) and the dynamics of various "shallow water" MHD waves in the solar tachocline have been studied by Schecter et al. (2001) (see also De Sterck 2001). Large-scale Rossby-like waves are absent from their consideration as the f-plane approximation has been used. However, we should mention the recent work by Leprovost & Kim (2007), which studies the influence of shear, Rossby, and Alfvén waves on the transport properties of MHD turbulence on a β-plane in the solar tachocline.

Here we use the MHD "shallow water" equations in order to study the influence of a toroidal magnetic field on the dynamics of Rossby waves in a rotating sphere. First we use Cartesian coordinates and derive the dispersion relation of magnetic Rossby waves in the β-plane approximation. Next, we solve the problem in spherical geometry, thus deriving the propagation properties of magnetic Rossby waves with a wavelength comparable to the radius of the sphere.

2. Basic considerations

Let us consider a thin layer of ideal conductive fluid on a sphere rotating with angular velocity $\mathbf{\Omega}_0$. The layer is taken as an incompressible fluid with a rigid base and a free upper surface and is permeated by a horizontal uniform magnetic field. The unperturbed uniform thickness of the layer, H_0, is smaller than the stratification scale height and so the medium density can be considered uniform. This system differs from the classical shallow water system only by the presence of the magnetic field. Then, the "shallow water" MHD equations in an inertial frame can be written as (Gilman 2000; Schecter et al. 2001)

$$\partial_t \boldsymbol{B} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{V}, \tag{1}$$

$$\partial_t \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = \frac{1}{4\pi\rho}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - g\nabla H, \tag{2}$$

$$\partial_t H + \nabla\cdot(H\boldsymbol{V}) = 0, \tag{3}$$

where $\boldsymbol{V}$ and $\boldsymbol{B}$ are the horizontal velocity and magnetic field, H is the thickness of the layer, ρ is the fluid density, ∇ is the horizontal gradient and g is the gravitational acceleration. The divergence-free condition for the magnetic field, which arises

from the requirement that $\boldsymbol{B}$ is parallel to the upper free surface, takes the form (Gilman 2000)

$$\nabla\cdot(H\boldsymbol{B}) = 0. \tag{4}$$

A general feature of the large-scale dynamics in such a system is that it does not significantly depend on the chosen geometry (flat, spherical or cylindrical) for equatorially trapped waves (Longuet-Higgins 1965; Pedlosky 1987). However, the consideration of spherical geometry is desirable for waves with wavelength comparable to the size of the sphere. Therefore, we first study the problem in the simpler Cartesian coordinates and then turn to the more complicate spherical geometry.

3. Cartesian coordinates

We consider a local Cartesian coordinate frame (x, y, z) in which the x axis is directed towards the rotation, the y axis is directed towards the north pole of the sphere and the z axis is directed vertically.

Let us next consider that the unperturbed magnetic field, $(B_x, 0, 0)$, is directed along the x axis. Then, after linearizing Eqs. (1)–(3) their components are written in the rotating frame as

$$\frac{\partial u_x}{\partial t} - f u_y = \frac{B_x}{4\pi\rho}\frac{\partial b_x}{\partial x} - g\frac{\partial h}{\partial x}, \tag{5}$$

$$\frac{\partial u_y}{\partial t} + f u_x = \frac{B_x}{4\pi\rho}\frac{\partial b_y}{\partial x} - g\frac{\partial h}{\partial y}, \tag{6}$$

$$\frac{\partial b_x}{\partial t} = B_x\frac{\partial u_x}{\partial x}, \quad \frac{\partial b_y}{\partial t} = B_x\frac{\partial u_y}{\partial x}, \tag{7}$$

$$\frac{\partial h}{\partial t} + H_0\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0, \tag{8}$$

where u_x, u_y, b_x and b_y are the velocity and magnetic field perturbations, $h = H - H_0$ is the perturbation of the layer thickness and $f = 2\Omega_0 \sin\Theta$ is the Coriolis parameter (with Θ the latitude). For zero magnetic field this system transforms into the HD shallow water equations (Pedlosky 1987).

Differentiation with respect to time of Eqs. (5)–(6) and using Eqs. (7)–(8) gives

$$\frac{\partial^2 u_x}{\partial t^2} - f\frac{\partial u_y}{\partial t} = v_A^2\frac{\partial^2 u_x}{\partial x^2} + C_0^2\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x\partial y}\right), \tag{9}$$

$$\frac{\partial^2 u_y}{\partial t^2} + f\frac{\partial u_x}{\partial t} = v_A^2\frac{\partial^2 u_y}{\partial x^2} + C_0^2\left(\frac{\partial^2 u_x}{\partial x\partial y} + \frac{\partial^2 u_y}{\partial y^2}\right), \tag{10}$$

where $v_A = B_x/\sqrt{4\pi\rho}$ and $C_0 = \sqrt{gH_0}$ are the Alfvén and surface gravity speeds, respectively.

We now perform a Fourier analysis of the form $\exp(-i\omega t + ik_x x)$ and after some algebra obtain

$$\frac{\partial^2 u_y}{\partial y^2} + \left[\frac{\omega^2}{C_0^2} - k_x^2 - \frac{k_x^2 v_A^2}{C_0^2} - \frac{\omega^2 f^2}{C_0^2(\omega^2 - k_x^2 v_A^2)} - \frac{k_x\omega}{(\omega^2 - k_x^2 v_A^2)}\frac{\partial f}{\partial y}\right]$$
$$\times u_y = 0. \tag{11}$$

When $v_A = 0$ this equation governs the linear dynamics of various kinds of waves (namely Poincaré, Kelvin and Rossby waves) in the HD shallow water approximation (Pedlosky 1987), but the inclusion of the magnetic field leads to the modification of the wave modes.

At a given latitude, Θ_0, one can perform a Taylor expansion of the Coriolis parameter and retain the lowest order latitudinal variation of f, which leads to (Pedlosky 1987; Gill 1982)

$$f = f_0 + \beta y, \tag{12}$$

where the parameter

$$\beta = \frac{2\Omega_0}{R_0}\cos\Theta_0 \tag{13}$$

(with R_0 the radius of the sphere) plays a major role in the so called β-plane approximation. Away from the equator $\beta y \ll f_0$ and therefore from Eq. (11) we readily get

$$\frac{\partial^2 u_y}{\partial y^2} + \left[\frac{\omega^2}{C_0^2} - k_x^2 - \frac{k_x^2 v_A^2}{C_0^2} - \frac{\omega^2 f_0^2}{C_0^2(\omega^2 - k_x^2 v_A^2)} - \frac{k_x\omega\beta}{(\omega^2 - k_x^2 v_A^2)}\right]$$
$$\times u_y = 0. \tag{14}$$

Thus, we can now perform a Fourier analysis of the form $\exp(ik_y y)$, which gives the dispersion relation

$$\omega^4 - [2k_x^2 v_A^2 + f_0^2 + C_0^2(k_x^2 + k_y^2)]\omega^2 - C_0^2 k_x\beta\omega$$
$$+ k_x^2 v_A^2[k_x^2 v_A^2 + C_0^2(k_x^2 + k_y^2)] = 0. \tag{15}$$

This dispersion relation contains high and low frequency branches, which respectively correspond to magneto-gravity waves and to Alfvén and Rossby waves. Note that for $\beta = 0$ this dispersion relation transforms into that of f-plane MHD "shallow water" waves (Schecter et al. 2001).

We next concentrate in the case of small Alfvén speed, i.e. $v_A \ll C_0$, such as corresponds to the interiors of solar-like stars. Then, the high frequency branch of Eq. (15) contains Poincaré waves, whose dispersion relation is (Pedlosky 1987)

$$\omega^2 = f_0^2 + C_0^2(k_x^2 + k_y^2), \tag{16}$$

while the low frequency branch yields the dispersion relation

$$\omega^2 + \frac{k_x\beta}{k_x^2 + k_y^2}\omega - k_x^2 v_A^2 = 0. \tag{17}$$

Note that the dispersion relation (17) was first obtained by Hide (1966) in the two dimensional case (see also Acheson & Hide 1973). This formula reveals some interesting properties. For short wavelengths, i.e. large k_x, the last term in Eq. (17) dominates over the second one, which leads to the solution

$$\omega = \pm k_x v_A. \tag{18}$$

This is the dispersion relation of pure Alfvén waves unaffected by rotation and propagating eastward and westward in the toroidal direction.

Nevertheless, for large-scale motions pure Alfvén waves no longer exist and instead we have Rossby waves modified by the magnetic field. For large wavelengths, i.e. small k_x, Eq. (17) has two different solutions. For the high frequency solution one can easily recover the dispersion relation of HD Rossby waves,

$$\omega \approx -\frac{k_x\beta}{k_x^2 + k_y^2}. \tag{19}$$

For the low frequency solution we have the dispersion relation

$$\omega \approx \frac{k_x v_A^2(k_x^2 + k_y^2)}{\beta}. \tag{20}$$

Hence, the horizontal magnetic field causes the splitting of ordinary large-scale Rossby waves into two modes propagating in opposite directions. The high frequency mode has the properties of HD Rossby waves and can be called *fast magnetic Rossby mode*. But, additionally, a lower frequency mode arises whose frequency is significantly smaller than that of pure Alfvén and Rossby waves at the same spatial scale. Due to its small frequency it can be called *slow magnetic Rossby mode* ("hydromagnetic-planetary waves" in Acheson & Hide 1973). The phase speed of the mode in the x direction depends on both the Alfvén speed and the β parameter,

$$v_{ph} = \frac{\omega}{k_x} = \frac{v_A^2(k_x^2 + k_y^2)}{\beta}. \tag{21}$$

The phase speed is different from Alfvén and Rossby phase speeds, which again indicates the different nature of this wave mode.

Numerical dispersion diagrams for the general dispersion relation (15), i.e. without assuming $v_A \ll C_0$, are presented in Fig. 1. The upper panel displays all wave solutions and shows that Poincaré waves are almost not affected by the magnetic field. The middle panel is a detailed view of the low frequency branch of the dispersion diagram. It is clearly seen that for small scales, i.e. for large k_x, magnetic Rossby waves tend to the Alfvén wave solutions (dashed lines), whereas for small k_x, i.e. for large spatial scales, the two modes behave differently: the solution with higher negative frequency corresponds to HD Rossby waves (triangles) and the low frequency solution, which differs from pure Rossby and Alfvén wave dispersion curves, is a new wave mode. Finally, the bottom panel shows the perfect fit between the solutions to Eq. (15) and the approximate dispersion relation for magnetic Rossby waves (circles), Eqs. (19)–(20), in the limit of small k_x.

The present consideration in Cartesian coordinates gives the basic properties of magnetic Rossby waves. Nevertheless, to study the dynamics of Rossby waves with spatial scales comparable to the radius of the sphere it is desirable to use spherical coordinates. Therefore, in the next section we study the same problem in spherical coordinates (r, θ, ϕ).

4. Spherical coordinates

Let us consider an unperturbed toroidal magnetic field B_ϕ. Then, the linearized form of Eqs. (1)–(3) can be rewritten in the rotating frame as

$$\frac{\partial u_\theta}{\partial t} - 2\Omega_0 \cos\theta u_\phi + \frac{g}{R_0}\frac{\partial h}{\partial \theta} - \frac{B_\phi}{4\pi\rho R_0 \sin\theta}\frac{\partial b_\theta}{\partial \phi}$$
$$+ 2\frac{B_\phi}{4\pi\rho R_0}\frac{\cos\theta}{\sin\theta}b_\phi = 0, \tag{22}$$

$$\frac{\partial u_\phi}{\partial t} + 2\Omega_0 \cos\theta u_\theta + \frac{g}{R_0 \sin\theta}\frac{\partial h}{\partial \phi} - \frac{b_\theta}{4\pi\rho R_0}\frac{\partial B_\phi}{\partial \theta}$$
$$- \frac{B_\phi}{4\pi\rho R_0 \sin\theta}\frac{\partial b_\phi}{\partial \phi} - \frac{B_\phi}{4\pi\rho R_0}\frac{\cos\theta}{\sin\theta}b_\theta = 0, \tag{23}$$

$$\frac{\partial h}{\partial t} + \frac{H_0}{R_0 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta u_\theta) + \frac{H_0}{R_0 \sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0, \tag{24}$$

$$\frac{\partial b_\theta}{\partial t} - \frac{B_\phi}{R_0 \sin\theta}\frac{\partial u_\theta}{\partial \phi} = 0, \tag{25}$$
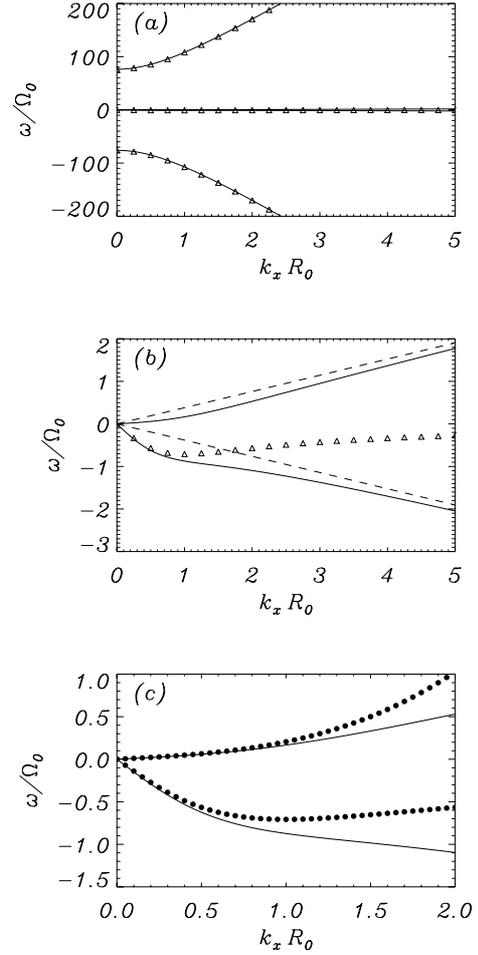

Fig. 1. a) Dispersion diagram of "shallow water" waves in the presence of a horizontal magnetic field. The two extreme upper and lower solutions correspond to Poincaré waves, which are almost unaffected by the magnetic field, whereas the two low frequency modes are magnetic Rossby waves. **b)** Low frequency branch of the dispersion diagram. **c)** Low frequency, large wavelength part of the dispersion diagram. In all panels solid lines correspond to the solutions to Eq. (15), triangles to the HD Poincaré and Rossby waves (i.e. in the absence of magnetic field), dashed lines to pure Alfvén waves (whose dispersion relation is given by Eq. (18)) and circles to the approximate analytical formulas (19)–(20). The parameters used to obtain the dispersion diagram are $k_y R_0 = 1$, $v_A/C_0 = 0.005$ and $\Theta = 45°$.

$$\frac{\partial b_\phi}{\partial t} + \frac{1}{R_0}\frac{\partial}{\partial \theta}(u_\theta B_\phi) = 0, \tag{26}$$

where u_θ, u_ϕ, b_θ and b_ϕ are the velocity and magnetic field perturbations, while $h = H - H_0$ is the perturbation of the layer thickness. For zero magnetic field this system transforms into the HD shallow water equations (Longuet-Higgins 1965).

We assume the unperturbed magnetic field to be $B_\phi = B_0 \sin\theta$, which means that it has a maximal value at the equator and tends to zero at the poles. We take a sinusoidal dependence of the magnetic field on θ for two main reasons: first, the sinusoidal profile simplifies the calculation in the spherical symmetry and second, the toroidal magnetic field seems to be located mainly in low latitudes due to the eruption of magnetic flux at these latitudes. Let us next introduce the new variables

$\hat{u}_\theta = \sin\theta u_\theta$, $\hat{u}_\phi = \sin\theta u_\phi$, $\hat{b}_\theta = \sin\theta b_\theta$, $\hat{b}_\phi = \sin\theta b_\phi$. Then, Eqs. (22)–(26) take the form:

$$\frac{\partial \hat{u}_\theta}{\partial t} - 2\Omega_0 \cos\theta \hat{u}_\phi + \frac{g}{R_0} \sin\theta \frac{\partial h}{\partial\theta} - \frac{B_0}{4\pi\rho R_0}\frac{\partial \hat{b}_\theta}{\partial\phi}$$
$$+2\frac{B_0}{4\pi\rho R_0}\cos\theta \hat{b}_\phi = 0, \qquad (27)$$

$$\frac{\partial \hat{u}_\phi}{\partial t} + 2\Omega_0 \cos\theta \hat{u}_\theta + \frac{g}{R_0}\frac{\partial h}{\partial\phi} - \frac{B_0}{4\pi\rho R_0}\frac{\partial \hat{b}_\phi}{\partial\phi}$$
$$-\frac{2B_0}{4\pi\rho R_0}\cos\theta \hat{b}_\theta = 0, \qquad (28)$$

$$\sin^2\theta\frac{\partial h}{\partial t} + \frac{H_0}{R_0}\sin\theta\frac{\partial \hat{u}_\theta}{\partial\theta} + \frac{H_0}{R_0}\frac{\partial \hat{u}_\phi}{\partial\phi} = 0, \qquad (29)$$

$$\frac{\partial \hat{b}_\theta}{\partial t} - \frac{B_0}{R_0}\frac{\partial \hat{u}_\theta}{\partial\phi} = 0, \qquad (30)$$

$$\frac{\partial \hat{b}_\phi}{\partial t} + \frac{B_0}{R_0}\sin\theta\frac{\partial \hat{u}_\theta}{\partial\theta} = 0. \qquad (31)$$

We now perform a Fourier analysis of the form $\exp(-i\omega t + is\phi)$ and define

$$\frac{\omega}{2\Omega_0} = \lambda, \quad \frac{4\Omega_0^2 R_0^2}{gH_0} = \epsilon, \quad \frac{v_A^2}{4\Omega_0^2 R_0^2} = \alpha^2, \quad \frac{gh}{2\Omega_0 R_0} = \eta,$$
$$\cos\theta = \mu, \quad -\sin\theta\frac{\partial}{\partial\theta} = (1-\mu^2)\frac{\partial}{\partial\mu} = D. \qquad (32)$$

After some algebra we get

$$-\lambda^2\tilde{u}_\theta - \mu\lambda\hat{u}_\phi - \lambda D\eta + s^2\alpha^2\tilde{u}_\theta + 2\alpha^2\mu D\tilde{u}_\theta = 0, \qquad (33)$$

$$-\lambda^2\hat{u}_\phi - \mu\lambda\tilde{u}_\theta + \lambda s\eta - 2s\alpha^2\mu\tilde{u}_\theta - \alpha^2 sD\tilde{u}_\theta = 0, \qquad (34)$$

$$\epsilon\lambda(1-\mu^2)\eta - D\tilde{u}_\theta - s\hat{u}_\phi = 0, \qquad (35)$$

where $\tilde{u}_\theta = i\hat{u}_\theta$.

Substitution of $\hat{u}_\phi$ from Eq. (34) into Eqs. (33) and (35) leads to

$$-\lambda^2\tilde{u}_\theta + \mu^2\hat{u}_\phi - (\lambda D + s\mu)\eta + s^2\alpha^2\tilde{u}_\theta + 2\alpha^2\mu D\tilde{u}_\theta$$
$$+\mu s\frac{\alpha^2}{\lambda}(D+2\mu)\tilde{u}_\theta = 0, \qquad (36)$$

$$\epsilon\lambda(1-\mu^2)\eta - D\tilde{u}_\theta + \frac{s}{\lambda}(\mu\tilde{u}_\theta - s\eta) + \frac{\alpha^2}{\lambda^2}s^2(D+2\mu)\tilde{u}_\theta = 0. \qquad (37)$$

After obtaining η from Eq. (37) and substituting it into Eq. (36) we get a single equation for $\tilde{u}_\theta$,

$$(\lambda D + s\mu)\left\{\frac{1}{s^2 - \epsilon\lambda^2(1-\mu^2)}\left[\lambda D - s\mu - \frac{\alpha^2}{\lambda^2}s^2\lambda(D+2\mu)\right]\right\}\tilde{u}_\theta$$
$$-(\lambda^2 - \mu^2)\tilde{u}_\theta + s^2\alpha^2\tilde{u}_\theta + 2\alpha^2\mu D\tilde{u}_\theta$$
$$+\mu s\frac{\alpha^2}{\lambda}(D+2\mu)\tilde{u}_\theta = 0. \qquad (38)$$

In the approximation for slowly rotating stars (such as in the solar case),

$$\frac{\epsilon}{s^2} = \frac{4\Omega_0^2 R_0^2}{gH_0 s^2} \ll 1 \qquad (39)$$

and Eq. (38) takes the form

$$[(\lambda D + s\mu)(\lambda D - s\mu) - \frac{\alpha^2}{\lambda}s^2(\lambda D + s\mu)(D+2\mu)$$
$$-s^2(\lambda^2 - \mu^2) + s^4\alpha^2 + 2\alpha^2 s^2\mu D + \mu s^3\frac{\alpha^2}{\lambda}(D+2\mu)]\tilde{u}_\theta = 0. \qquad (40)$$

The approximation (39) implies that magneto gravity waves (i.e. magnetic Poincaré and Kelvin waves) are neglected from our consideration, thus retaining only magnetic Rossby waves.

Now, Eq. (40) can be rewritten as

$$\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} - \frac{s^2}{1-\mu^2} + n(n+1)\right]\tilde{u}_\theta = 0, \qquad (41)$$

if

$$n(n+1) = -\frac{s\lambda + 2s^2\alpha^2}{\lambda^2 - \alpha^2 s^2}. \qquad (42)$$

Equation (41) is the associated Legendre differential equation (Abramowitz & Stegun 1964), whose typical solutions are the associated Legendre polynomials,

$$\tilde{u}_\theta = P_n^s(\cos\theta) \qquad (43)$$

if n is an integer (when n is not integer the solutions are the associated Legendre functions).

Equation (42) defines the dispersion relation for spherical magnetic Rossby waves,

$$n(n+1)\left(\frac{\lambda}{s}\right)^2 + \frac{\lambda}{s} + \alpha^2[2 - n(n+1)] = 0, \qquad (44)$$

where n plays the role of the poloidal wavenumber.

In the non-magnetic case, i.e. for $\alpha = 0$, the dispersion relation reduces to the HD Rossby mode solution (Longuet-Higgins 1965). But the magnetic field causes the splitting of the ordinary HD mode into fast and slow magnetic Rossby modes as in the rectangular case (it is worth recalling that Rossby-type waves are sometimes called planetary waves, which is probably a more appropriate name for the spherical geometry).

Before studying the dispersion diagram, let us first consider the properties of wave modes for particular values of n. For purely toroidal propagation, i.e. when $n = 0$, we have only one mode with the dispersion relation

$$\frac{\omega}{s} = -\frac{v_A^2}{\Omega_0 R_0^2}. \qquad (45)$$

In the weak magnetic field limit this mode has significantly lower frequency than pure Rossby and Alfvén waves and can be associated to the slow magnetic Rossby mode. Note, that $n = 0$ mode is absent in non-magnetic spherical case (Longuet-Higgins 1965).

For $n = 1$ we have only the fast magnetic Rossby mode, which in this case is identical to the HD Rossby mode, with the dispersion relation (Longuet-Higgins 1965)

$$\frac{\omega}{s} = -\Omega_0. \qquad (46)$$

For $n > 1$ we have both fast and slow magnetic Rossby modes. In the weak magnetic field limit, i.e. for $v_A \ll 2\Omega_0 R_0$, the dispersion relation for the fast magnetic Rossby mode is

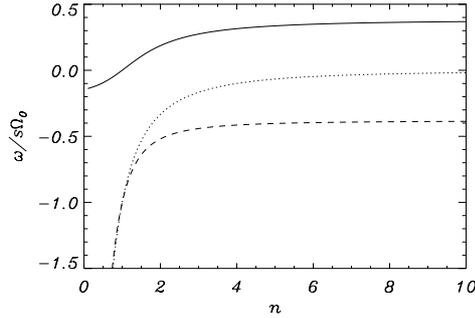$$\frac{\omega}{s} \approx -\frac{2\Omega_0}{n(n+1)} \qquad (47)$$

Fig. 2. Numerical dispersion diagram of spherical "shallow water" waves in the presence of a toroidal magnetic field. The dependence of the wave frequency on the poloidal wavenumber n is plotted here for $\alpha^2 = 0.036$. The continuous and dashed lines are the solutions for the slow and fast magnetic Rossby modes, respectively. The dotted line is the HD Rossby mode, which is obtained from Eq. (44) with $\alpha^2 = 0$.

Fig. 3. Frequencies of the $s = 1, n = 2$ harmonics of fast (dashed) and slow (continuous) magnetic Rossby modes vs. the ratio of the Alfvén speed to the rotation rate, $\alpha = v_A/2\Omega_0 R_0$. When α tends to zero, then the slow magnetic Rossby mode has a very low frequency, while the fast mode frequency tends to $\sim 0.3\Omega_0$.

(which is similar to HD Rossby mode, Longuet-Higgins 1965) and the dispersion relation for the slow magnetic Rossby mode is

$$\frac{\omega}{s} \approx -\frac{v_A^2}{2\Omega_0 R_0^2}[2 - n(n + 1)]. \tag{48}$$

Equations (47)–(48) show that the toroidal magnetic field causes the splitting of ordinary HD Rossby waves into two different modes propagating in opposite directions for $n > 1$. Such splitting is also present for all non-integer values of n smaller than 1. Thus, the general behaviour of the wave modes in spherical geometry is similar to that of the Cartesian case. There are some differences, however. First, the dispersion relation of magnetic Rossby waves in a Cartesian frame depends on the latitude; second, magnetic Rossby waves are dispersive with respect to the toroidal wavenumber, k_x, in Cartesian geometry (see Eqs. (19)−(20)), while they are not dispersive with respect to the toroidal wavenumber, s, in spherical geometry.

Numerical solutions to the general dispersion relation (44) are presented in Fig. 2. The dependence of the wave frequency on the poloidal wavenumber n is plotted here for $\alpha^2 = 0.036$. The continuous and dashed lines are the solutions for slow and fast magnetic Rossby modes, respectively. The dotted line is the HD Rossby mode, which is obtained from Eq. (44) with $\alpha^2 = 0$. It is clearly seen that for small scales, i.e. for large n, fast and slow magnetic Rossby waves behave similarly and tend to the Alfvén-like wave solutions, whereas for $n < 3$, i.e. for large spatial scales, the two modes behave differently: the solution with higher negative frequency corresponds to HD Rossby waves (dashed line) and the low frequency (continuous line) solution, which differs from pure Rossby wave dispersion curves, is a new wave mode (see similar consideration by Hide 1966; Acheson & Hide 1973). It must be mentioned that the value of the toroidal wavenumber, s, does not influence the general behaviour of the modes and that increasing s leads only to a linear increase of the frequency, ω.

The frequency of particular harmonics depends on the strength of the toroidal magnetic field. The frequency of the $s = 1, n = 2$ harmonics of fast (dashed) and slow (continuous) magnetic Rossby modes vs. the ratio of the Alfvén speed to the rotation rate, $\alpha = v_A/2\Omega_0 R_0$, are plotted in Fig. 3. The fast and slow modes have lower frequency than the angular velocity, Ω_0, for $\alpha < 0.5$ and $\alpha < 0.8$ respectively. When α goes to zero, the fast mode frequency tends to $0.3\Omega_0$, while the slow mode frequency tends to zero. Therefore, the slow magnetic Rossby mode may have very low frequency depending on the ratio of the Alfvén speed to the angular velocity.
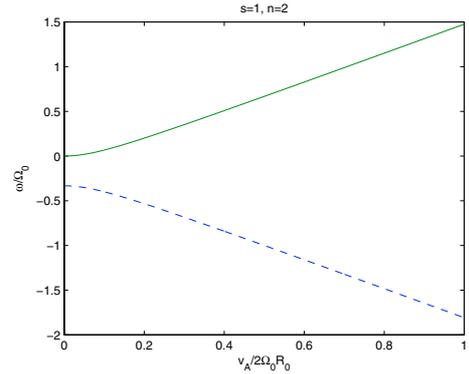
5. Discussion

The recently developed "shallow water" magnetohydrodynamic approximation (Gilman 2000) has stimulated further study of MHD wave modes in this system. While ordinary "shallow water" modes (Poincaré, Kelvin, Rossby) have been intensively studied in the geophysical context (Pedlosky 1987; Gill 1982), the inclusion of a horizontal magnetic field enriches the wave spectrum. Magneto-gravity and Alfvén modes in the "shallow water" MHD system have been recently studied by Schecter et al. (2001). However, large-scale modes (those with stellar spatial dimensions; for example, Rossby waves) were absent from their consideration due to the use of the f-plane approximation. On the contrary, here we emphasize the large-scale behavior of wave modes in the "shallow water" system. Considering the β-plane approximation and, especially, spherical coordinates, enables us to study the wave dynamics on spatial scales corresponding to stellar dimensions. Particular attention is paid to the slow magnetic Rossby mode, expressed by Eqs. (20), (45), (48). The presence of this mode in rotating magnetised fluids was first pointed out by Hide (1966) using the two dimensional Cartesian β-plane approximation in the context of the Earth's liquid core. Here we derive analytical dispersion relations of this mode in both Cartesian and spherical "shallow water" MHD systems. In the low Alfvén speed limit (compared to the surface gravity speed), this mode has a smaller frequency than that of pure Alfvén and Rossby modes and consequently may have new interesting consequences in large-scale stellar dynamics.

However, this consideration needs some modifications when applied to concrete astrophysical situations; for example, to the solar tachocline. It is believed that the tachocline is divided in two parts: the inner "radiative" layer with a strongly stable stratification and the outer "overshoot" layer with a weakly stable stratification (Gilman 2000). The subadiabatic stratification provides negative buoyancy in both layers, which leads to the so-called "reduced gravity", g_r (Gilman 2000; Schecter et al. 2001). Therefore, the developed theory of fast and slow magnetic Rossby waves should be modified for tachoclines of solar-like stars using the reduced gravity instead of the ordinary one. Then the results can be quite different for the radiative and overshoot layers due to the significant difference between the reduced

gravity there. Schecter et al. (2001) estimated the reduced gravity in the radiative layer as $500-1.5 \times 10^4$ cm s^{-2} and in the overshoot layer as $0.05-5$ cm s^{-2}. Then the surface gravity speed C_0 is higher than the Alfvén speed in the radiative layer, but not in the overshoot one, where both speeds may have similar values (Schecter et al. 2001). Therefore, the results obtained here can be easily applied to the radiative part of tachocline, but not to the overshoot one. The exception is the dispersion relation (15), which can be applied to both parts of the tachocline.

It must be also mentioned that differential rotation, typical of the tachocline dynamics, is absent from our consideration. The goal of this paper is to study the influence of the magnetic field on the large-scale dynamics of a "shallow water" system in general. However, differential rotation should be taken into account in more realistic models of wave dynamics in the tachocline.

6. Conclusions

The influence of a toroidal magnetic field on the dynamics of Rossby waves in a thin layer of ideal conductive fluid on a rotating sphere is studied in the shallow water MHD approximation for both, Cartesian and spherical geometries. It is shown that in both cases the magnetic field causes the splitting of low order (long wavelength) ordinary Rossby wave harmonics into two modes (here called magnetic Rossby modes). The high frequency mode (the fast magnetic Rossby mode) corresponds to ordinary HD Rossby waves slightly modified by the magnetic field, while the low frequency solution leads to a new mode (the slow magnetic Rossby mode) with interesting properties. Low order (with respect to the poloidal wavenumber) harmonics of the slow magnetic Rossby mode have lower frequency than the pure Rossby and Alfvén wave frequencies of the corresponding harmonics.

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