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Evolution of low-frequency features in the CMB spectrum due to stimulated Compton scattering and Doppler broadening

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ABSTRACT

We discuss a new analytic solution of the Kompaneets equation for physical situations in which low frequency photons, forming relatively narrow spectral details, are Compton scattered in an isotropic, infinite medium with an intense ambient blackbody field that is very close to full thermodynamic equilibrium with the free electrons. In this situation the *background-induced stimulated Compton scattering* slows down the motion of photons toward higher frequencies by a factor of 3 in comparison with the solution that only takes Doppler broadening and boosting into account. This new solution is important for detailed computations of cosmic microwave background spectral distortions arising from uncompensated atomic transitions of hydrogen and helium in the early Universe. It also clearly shows that the broadening of weak lines in this situation only depends on the Compton y-parameter defined by $T_{\rm e}$, even though the evolution of the ambient CMB blackbody spectrum itself is described by $y \propto T_{\rm e} - T_{\gamma}$. In addition, we derive another analytic solution that only includes the background-induced stimulated Compton scattering and is valid for power law ambient radiation fields. This solution might have interesting applications for radio lines arising inside of bright extra-galactic radio sources, where according to our estimates line shifts because of background-induced stimulated scattering could be amplified and even exceed the line broadening due to the Doppler effect.

Key words. cosmology: theory – cosmology: cosmic microwave background – line: formation – radiative transfer – scattering

1. Introduction

In astrophysical environments one often encounters the situation that narrow low-frequency emission or absorption features are scattering off free electrons in the presence of a much brighter and broad ambient background radiation field. The clearest example of such a setting is given after the cosmological recombination of He III ($z \sim 6000$) in our Universe. During this epoch, recombinational lines were produced that today should be observable in the Rayleigh-Jeans part of the cosmic microwave background (CMB) spectrum, at dm and cm wavelengths (Dubrovich & Stolyarov 1997; Rubiño-Martín et al. 2008; Sunyaev & Chluba 2008). Also, the probability of Thomson scattering was still very high, so that these photons underwent multiple scatterings in the presence of the much more intense CMB before they eventually reach an observer.

Another obvious example can be found in the pre-recombinational epoch of hydrogen and helium *if* there was some early energy release (Chluba & Sunyaev 2008). It is well-known that any energy release below $z \sim 5 \times 10^4$ will lead to a characteristic broad y-type distortion (Zeldovich & Sunyaev 1969) of the background radiation. In this situation, because of the non-blackbody CMB photon field, uncompensated atomic loops would develop (Lyubarsky & Sunyaev 1983), which at high redshifts start with the capture of a free electron by a proton or He III ion to an excited state, and end with the absorption of a high frequency photon (mostly in the H I or He II Lyman-continuum), again producing a free electron and an ion. In between, the captured electron may cascade towards lower levels, releasing several photons. This process is trying to help restoring full thermodynamic equilibrium after the energy injection has occurred. It results in the release of several photons per one absorbed high-frequency photon, and leads to additional, relatively narrow, low-frequency spectral features in the CMB, but in this case from redshifts *prior* to the actual recombination epoch. Also, their amplitude and phase dependence should differ from those of the CMB distortions arising during He III → He II-recombination (Chluba & Sunyaev 2008).

It is clear that in both cases the multiple scattering of photons by hot electrons should lead to broadening and shifting of these spectral features. Here we investigate the evolution of narrow lines due to both *stimulated scattering* in the presence of the much brighter CMB with temperature $T_{\gamma} = 2.7\,(1 + z)\,$K, and *Doppler broadening* and *boosting*. According to our analytical solutions, in this situation the background-induced stimulated Compton scattering at low frequencies slows down the motion of photons toward higher frequencies by a factor of 3 in comparison with the solution that only takes Doppler broadening and boosting (Zeldovich & Sunyaev 1969) into account.

We present two analytical solutions demonstrating this fact. In the first we only include the pure effect of stimulated scattering in the presence of a bright power law ambient photon field, neglecting the Doppler effect (see Sect. 3), while in the second solution we take both effects into account simultaneously, but restrict ourselves to the case of a low-frequency blackbody ambient photon field with temperature $T_{\gamma} \sim T_{\rm e}$, where $T_{\rm e}$ denotes the electron temperature (see Sect. 4). The second solution also clearly shows that the broadening of weak lines in this situation only depends on the y-parameter defined by $T_{\rm e}$, even though the evolution of the ambient CMB blackbody spectrum itself is described by $y \propto T_{\rm e} - T_{\gamma}$.

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2. Summary of previous analytic solutions to the Kompaneets equation

The repeated Compton scattering of photons by thermal electrons in isotropic, infinite media can be described using the well-known Kompaneets equation (Kompaneets 1956)

$$\frac{\partial n(x_e, y_e)}{\partial y_e} = \frac{1}{x_e^2}\frac{\partial}{\partial x_e} x_e^4 \left[\frac{\partial n}{\partial x_e} + n(1+n)\right], \quad (1)$$

where $n(x_e, y_e) = c^2 I_\nu / 2h\nu^3$ is the photon occupation number, and $y_e = \int \frac{kT_e}{m_e c^2} N_e \sigma_T c \, dt$ the Compton y-parameter. In addition, $x_e = h\nu/kT_e$ is the dimensionless frequency, where T_e denotes the electron temperature and N_e the free electron number density.

As is well understood, the first term in brackets ($\propto \partial_{x_e} n$) describes the *diffusion* of photons along the frequency axis, the second term ($\propto n$) the motion of photons towards low frequencies due to the *recoil* effect, and the last term ($\propto n^2$) the effect of *stimulated scattering*, which physically is also related to recoil (e.g. see Sazonov & Sunyaev 2000). Equation (1) has been studied in great detail, both numerically (e.g. Pozdniakov et al. 1983) and analytically in several limiting cases, but, to our knowledge, no general analytic solution has been found.

In situations when the recoil term and stimulated scatterings are not important (i.e. $n[n+1] \ll \partial_{x_e} n$), Zeldovich & Sunyaev (1969) gave a solution for arbitrary value of y_e, which reads

$$n(\nu, y_e)|_{\rm diff} = \frac{1}{\sqrt{4\pi\, y_e}} \int n_0(\tilde{\nu})\, e^{-\frac{(\ln[\nu/\tilde{\nu}] + 3y_e)^2}{4 y_e}} \frac{d\tilde{\nu}}{\tilde{\nu}}, \quad (2)$$

where $n_0(\nu)$ is the initial ($y_e = 0$) photon occupation number at frequency ν. For $y_e \ll 1$ the broadening of an initially narrow line is given by $\Delta\nu/\nu \sim \pm 2\sqrt{y_e \ln 2}$. Due to Doppler boosting, the maximum of the specific intensity I_ν is moving along $\nu_{\rm line}(y_e) = \nu_0\, e^{3y_e}$, so that, for $y_e \ll 1$, one has $\Delta\nu/\nu \sim 3y_e$, implying that photons on average are upscattered.

In the case where the diffusion term and stimulated scatterings are not important, Arons (1971) and independently Illarionov & Syunyaev (1972), using a different mathematical approach, gave a solution that can be written in the form

$$n(\nu, \tau_T)|_{\rm recoil} = \frac{1}{[1-\omega\,\tau_T]^4}\, n_0\left(\frac{\nu}{1-\omega\,\tau_T}\right). \quad (3)$$

Here $\omega = \frac{h\nu}{m_e c^2}$ and $\tau_T = N_e \sigma_T c\, t$ is the Thomson-scattering optical depth. It was assumed here that N_e is independent of time, but it is easy to write the solution for any dependence of N_e on time or redshift. Equation (3) simply describes the motion of photons towards lower frequencies, where for the initial photon distribution, $n_0(\nu) = A\,\delta(\nu - \nu_0)/\nu^2$, later the line is located at $\nu_{\rm line}(\tau_T) = \nu_0/[1 + \omega_0 \tau_T]$. For $\omega\tau_T \ll 1$ the line shift due to recoil is given by $\Delta\nu/\nu \sim -\omega\tau_T$. This shows that at low frequencies ($h\nu \ll kT_e$) the recoil effect can be neglected in comparison with Doppler boosting. Also one should mention that, for any fixed frequency ν, those photons initially located at $\nu_1 > \nu$ will cross this frequency at time $\tau_T \geq 1/\omega - 1/\omega_1$. This also implies that, for $\tau_T = 1/\omega$, those photons initially coming from very high frequencies (i.e. in the limit $\nu_1 \to \infty$) would have reached ν. For any *physical* initial photon distribution, $n_0(\nu')$, which is defined on the interval $\nu' \in [0, \infty)$, the solution $n(\nu, \tau_T)|_{\rm recoil}$ therefore vanishes at $\tau_T \geq 1/\omega$ and given ν.

The third analytic solution was found for the case when only the stimulated scattering term is important (Zeldovich & Levich 1968; Sunyaev 1970). The non-linear nature of this problem can lead to the appearance of *shock waves* in the photon field, e.g. as explained in Zeldovich & Levich (1968) and Zeldovich & Syunyaev (1972). Here the solution is determined by the implicit equation

$$\nu = \phi(f) - 2\frac{h}{m_e c^2}\tau_T\, f, \quad (4)$$

with $f(\nu, \tau_T) = \nu^2\, n(\nu, \tau_T)$ where $\phi(z)$ can be found from the initial condition ($\phi(z) \equiv f_0^{-1}(z)$, where $f_0^{-1}(z)$ is the inverse function of $f(\nu, \tau_T)$ at $\tau_T = 0$).

The last two solutions do not depend on the temperature of the electrons, since physically in both cases the motion of the electrons in lowest order does not play any role.

3. Solution for purely background-induced stimulated scattering

In this section we discuss the situation when photons from a low-frequency spectral feature are scattering off thermal electrons at temperature T_e within an intense photon background, with photon occupation number, $n_{\rm back}(\nu) \gg 1$. If we neglect the evolution of the background photon field[1] and assume that recoil and Doppler broadening are negligible, then, inserting $n = n_{\rm back} + \Delta n$ into Eq. (1), one is left with

$$\frac{\partial \Delta n(\nu, \tau_T)}{\partial \tau_T} \approx \frac{2}{\nu^2}\frac{h}{m_e c^2}\frac{\partial}{\partial \nu}\nu^4\, n_{\rm back}(\nu)\, \Delta n\,. \quad (5)$$

This equation describes the purely background-induced motion of photons along the frequency axis. Due to the large factor $2\,n_{\rm back} \gg 1$, the speed of this motion is increased, so that the line shift can still be significant even for $2\frac{h\nu}{m_e c^2}\tau_T \ll 1$.

According to Eq. (5) photons are moving along the characteristic

$$\tau_T + \int \frac{d\nu'}{2\nu'\, n_{\rm back}(\nu')\, \frac{h\nu'}{m_e c^2}} \equiv \tau_T + \int \frac{d\nu'}{2\frac{kT_b(\nu')}{m_e c^2}\,\nu'}, \quad (6)$$

where we have also introduced the brightness temperature

$$T_b(\nu) = \frac{c^2 I_\nu^{\rm back}}{2k\nu^2} \equiv \frac{h\nu}{k}\, n_{\rm back}(\nu) \quad (7)$$

of the photon field.

With Eq. (6), it is in principle possible to write down the solution of Eq. (5) for general background radiation fields. However, here we now focus on background radiation fields with power law spectral dependence, i.e. $n_{\rm back}(\nu) \propto \nu^\beta$, where β denotes the power law index[2]. Then, given the initial photon distribution $\Delta n_0(\nu) = \Delta n(\nu, \tau_T = 0)$ of a spectral feature, one can readily solve Eq. (5) using Eq. (6).

Below we now discuss the solution for the three cases $\beta = -1$ (Sect. 3.1), $\beta > -1$ (Sect. 3.2), and $\beta < -1$ (Sect. 3.3) separately. The first case is the most interesting for us in the context of the recombinational spectral features in the Rayleigh-Jeans part of the CMB. The second case may be relevant for self-absorbed synchrotron radio sources, having $\alpha = 2.5$ and $\beta = -0.5$. The classical example is a Wien spectrum with $\beta = 0$ at $h\nu \ll kT_e$. In the non-linear problem, such spectra might lead to the appearance of a shock wave in the radiation spectrum because high-frequency photons move faster than low-frequency ones. The third case is applicable to the typical spectra of extra-galactic synchrotron radio sources having $\alpha = -0.7$ and $\beta = -3.7$.

[1] This assumption is justified, for example, when the frequency shifts of the background radiation field are relatively small.

[2] When the spectral intensity scales like $I_\nu \propto \nu^{-\alpha}$, one has $\beta = -\alpha - 3$. For the Rayleigh-Jeans part of a blackbody spectrum $\beta = -1$.

3.1. Rayleigh-Jeans limit ($\beta = -1$)

For $\beta = -1$, the brightness temperature of the ambient radiation field becomes frequency-independent and equal to the Rayleigh-Jeans temperature, $T_{\rm b} \equiv T_{\rm RJ}$. Then Eq. (5) can be cast into the form $\partial_{\tau_{\rm T}} \Delta n(\nu, \tau_{\rm T}) = \frac{2}{\nu^2}\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\,\partial_\nu\, \nu^3\, \Delta n$, where photons are moving along the characteristic $\tau_{\rm T} + \ln(\nu)/2\,\frac{kT_{\rm RJ}}{m_{\rm e}c^2}$. With the replacements $s = \nu^3\,\Delta n$ and $\xi = \ln(\nu)$, one then finds

$$\Delta n(\nu, \tau_{\rm T}) = {\rm e}^{6\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\tau_{\rm T}}\,\Delta n_0\left(\nu\,{\rm e}^{2\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\tau_{\rm T}}\right). \quad (8)$$

For $\Delta n_0(\nu_0) = A\,\delta(\nu-\nu_0)/\nu^2$, at time $\tau_{\rm T}$ the line will be centered at $\nu^\delta(\tau_{\rm T}) = \nu_0\,{\rm e}^{-2\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\tau_{\rm T}}$. Assuming $2\,\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\tau_{\rm T} \ll 1$ one then has $\Delta\nu/\nu_0 \sim -2\,\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\tau_{\rm T}$, showing that the relative line shift is independent of frequency. For $T_{\rm RJ} \equiv T_{\rm e}$ this expression indicates that, because of stimulated scattering of photons within the Rayleigh-Jeans part of a blackbody ambient radiation field, they are moving with a speed $\propto 2\,\frac{kT_{\rm RJ}}{m_{\rm e}c^2}\tau_{\rm T} \equiv 2y_{\rm e}$ towards lower frequencies. This is 3/2 times lower than the line shift due to Doppler boosting (see Sect. 2), but is directed in the opposite direction. Below we discuss this case in more detail, including the broadening of the line due to the Doppler effect (see Sect. 4).

3.2. Case $\beta > -1$

For arbitrary β, the general solution of Eq. (5) is given by[3]

$$\Delta n(\nu, \tau_{\rm T}) = \frac{\Delta n_0\left(\frac{\nu}{[1-2(\beta+1)\,n_{\rm back}(\nu)\,\omega\,\tau_{\rm T}]^{\frac{1}{\beta+1}}}\right)}{[1-2(\beta+1)\,n_{\rm back}(\nu)\,\omega\,\tau_{\rm T}]^{\frac{\beta+4}{\beta+1}}}. \quad (9)$$

First focusing on the case $\beta > -1$ (i.e. $\beta+1 > 0$), as explained for the pure recoil case (see Sect. 2 for details), at fixed ν the solution $\Delta n(\nu, \tau_{\rm T}) \equiv 0$ at times $\tau_{\rm T} \geq [2(\beta+1)\,n_{\rm back}(\nu)\,\omega]^{-1}$.

For initial photon distribution, $\Delta n_0(\nu) = A\,\delta(\nu-\nu_0)/\nu^2$, the line is located at $\nu^\delta(\tau_{\rm T}) = \nu_0/[1+2(\beta+1)\,n_{\rm back}(\nu_0)\,\omega_0\,\tau_{\rm T}]^{\frac{1}{\beta+1}}$. Under the condition $2(\beta+1)\,n_{\rm back}(\nu_0)\,\omega_0\,\tau_{\rm T} \ll 1$, one then has $\Delta\nu/\nu_0 \sim -2\,n_{\rm back}(\nu_0)\,\omega_0\,\tau_{\rm T}$. This shows that independent of β, photons are *always* moving toward lower frequencies, where the speed of this motion is increased by $2\,n_{\rm back} \gg 1$, as compared to the pure recoil case (see Sect. 2). As in the case of pure recoil for $\beta > -1$, the motion of spectral features is faster at higher frequencies.

3.3. Case $\beta < -1$

Using Eq. (9) for $\beta < -1$ (i.e. $\beta+1 < 0$ and $|\beta+1| > 0$), one finds

$$\Delta n(\nu, \tau_{\rm T}) = \frac{\Delta n_0\left(\nu\,[1+2|\beta+1|\,n_{\rm back}(\nu)\,\omega\,\tau_{\rm T}]^{\frac{1}{|\beta+1|}}\right)}{[1+2|\beta+1|\,n_{\rm back}(\nu)\,\omega\,\tau_{\rm T}]^{-\frac{\beta+4}{|\beta+1|}}}. \quad (10)$$

One can see that in contrast to the pure recoil case (Sect. 2) and the background induced case for $\beta > -1$ (Sect. 3.2), here photons initially coming from very high frequencies (i.e. in the limit $\nu_1 \to \infty$) reach ν only in the limit $\tau_{\rm T} \to \infty$.

For initial photon distribution, $\Delta n_0(\nu) = A\,\delta(\nu-\nu_0)/\nu^2$, the line is later located at $\nu^\delta(\tau_{\rm T}) = \nu_0\,[1-2|\beta+1|\,n_{\rm back}(\nu_0)\,\omega_0\,\tau_{\rm T}]^{\frac{1}{|\beta+1|}}$. This shows that photons starting at ν no longer asymptotically (i.e. $\tau_{\rm T} \to \infty$) approach $\nu \to 0$, as in the cases $\beta \geq -1$, but formally 'reach' it at $\tau_{\rm T} = [2|\beta+1|\,n_{\rm back}(\nu)\,\omega]^{-1}$. Also for the condition $2|\beta+1|\,n_{\rm back}(\nu_0)\,\omega_0\,\tau_{\rm T} \ll 1$, one has $\Delta\nu/\nu_0 \sim -2\,n_{\rm back}(\nu_0)\,\omega_0\,\tau_{\rm T}$, so that, again independent of β, photons are moving toward lower frequencies. In contrast to the pure recoil case, the motion of spectral features is faster at lower frequencies.

[3] From this expression the solution Eq. (8) can also be obtained by taking the limit $\beta \to -1$.

3.4. Line shift in terms of the brightness temperature

In all cases of β it is possible to rewrite the expression for the line shift in terms of the brightness temperature, Eq. (7), of the background field in the vicinity of the spectral feature. For $\Delta\nu/\nu \ll 1$ this yields

$$\left.\frac{\Delta\nu}{\nu}\right|_{\rm back-ind} \sim -2\,\frac{kT_{\rm b}(\nu)}{m_{\rm e}c^2}\tau_{\rm T}. \quad (11)$$

For $T_{\rm b} > \frac{3}{2}T_{\rm e}$, it is clear that the line shift due to background-induced stimulated scattering exceeds the shift towards higher frequencies due to Doppler boosting. Therefore the net motion of the line center can be directed towards lower frequencies, even if one includes the Doppler term.

Comparing with the case of pure Doppler broadening (see Sect. 2), it is also clear that, for $y_{\rm e} \ll 1$ and $y_{\rm e} \gtrsim 4\,[T_{\rm e}/T_{\rm b}(\nu)]^2\ln 2$, the background-induced line shift becomes larger than the FWHM Doppler broadening connected with the diffusion term. In this situation one can neglect the effect of line broadening due to the Doppler term, and, according to Eq. (11), for small Thomson optical depth still obtains a rather significant line shift. In radio spectroscopy is possible to determine tiny shifts or broadening in the frequency of narrow spectral features (for example of 21 cm lines from high-redshift radio galaxies).

3.5. Evolution of the background field and the relative velocity of a spectral feature for small line shifts

In the derivation given above we have neglected the evolution of the background radiation field. This is possible when the line shifts $\Delta\nu/\nu$ are small. If we insert $n = n_{\rm back} \propto \nu^\beta$ into the Kompaneets Eq. (1) and assume that $y_{\rm e}$ is sufficiently small, then we can directly write the solution for the change of the photon occupation number as

$$\frac{\Delta n_{\rm back}}{n_{\rm back}} = y_{\rm e}\,\{\beta(3+\beta) + (4+\beta)x_{\rm e} + 2n_{\rm back}(2+\beta)x_{\rm e}\}. \quad (12)$$

With this one can now find the corresponding overall shift of the background radiation field at initial frequency $x_{\rm e,0}$, solving the equation $n_{\rm back}(x_{\rm e,0}) \equiv n_{\rm back}(x_{\rm e})[1+\frac{\Delta n_{\rm back}(x_{\rm e})}{n_{\rm back}(x_{\rm e})}]$, with $x_{\rm e} = x_{\rm e,0}(1+\delta)$, for $\delta \equiv -\Delta\nu/\nu$. Under the condition $\beta \neq 0$ one finds

$$\left.\frac{\Delta\nu}{\nu}\right|_{\rm back} \sim \frac{2\,(2+\beta)}{\beta}\frac{kT_{\rm b}(\nu)}{m_{\rm e}c^2}\tau_{\rm T} + \frac{4}{\beta}\frac{h\nu}{m_{\rm e}c^2}\tau_{\rm T} + (3+\beta+x_{\rm e})\frac{kT_{\rm e}}{m_{\rm e}c^2}\tau_{\rm T}. \quad (13)$$

With $|\Delta\nu/\nu| \ll 1$, one can give the condition under which it is possible to neglect the evolution of the background field.

It is important to mention that Eq. (12) does only conserve a Wien spectrum ($n = {\rm e}^{-x_{\rm e}} \approx 1$ for $x_{\rm e} \ll 1$) and a Rayleigh-Jeans spectrum ($n = 1/x_{\rm e}$) to order $x_{\rm e}$. This is because higher order terms cannot be taken into account by simple power laws. Only $n = 1/[{\rm e}^{x_{\rm e}+\mu} - 1]$ with constant μ and $T_{\rm e} = T_\gamma$ is truly invariant.

4. Scattering of low-frequency photons by free electrons in an intense ambient blackbody field with $T_\gamma \equiv T_e$

If we assume that at $y_e = 0$ the initial photon field is given by $n = n_{pl}(x_e)+\Delta n$, where $n_{pl}(x) = 1/[e^x-1]$ is the blackbody occupation number, $x_e = h\nu/kT_e$, and Δn is a small ($\Delta n/n_{pl}(x_e) \ll 1$) spectral distortion, then with $\partial_{x_e} n_{pl}(x_e) \equiv -n_{pl}(x_e)[1 + n_{pl}(x_e)]$ and $\partial_{y_e} n_{pl}(x_e) = 0$, from Eq. (1) one has

$$\frac{\partial \Delta n(x_e, y_e)}{\partial y_e} = \frac{1}{x_e^2} \frac{\partial}{\partial x_e} x_e^4 \left[\frac{\partial \Delta n}{\partial x_e} + \Delta n(1 + 2n_{pl}(x_e) + \Delta n)\right]. \quad (14)$$

Note that $\partial_{y_e} n_{pl}(x_e) = 0$ is fulfilled when $T_\gamma \equiv T_e$. If we now neglect terms of $O(\Delta n^2)$ and assume that $x_e \ll 1$, and hence $n_{pl}(x_e) \approx 1/x_e \gg 1$, then Eq. (14) reads as

$$\frac{\partial \Delta n(x_e, y_e)}{\partial y_e} = \frac{1}{x_e^2} \frac{\partial}{\partial x_e} x_e^4 \left[\frac{\partial \Delta n}{\partial x_e} + 2\frac{\Delta n}{x_e}\right]. \quad (15)$$

Introducing the variables $\xi = \ln(x_e)$ and $s = x_e^3 \Delta n$, we then have $\partial_{y_e} s(\xi, y_e) = \partial_\xi^2 s - \partial_\xi s$. Similar to Zeldovich & Sunyaev (1969), we now transform to $z = \xi - y_e$ so that this equation reduces to the normal diffusion equation $\partial_{y_e} s = \partial_z^2 s$. Therefore the solution of Eq. (15) is

$$\Delta n(\nu, y_e) = \frac{1}{\sqrt{4\pi\, y_e}} \int \frac{\tilde{\nu}^3}{\nu^3}\, \Delta n(\tilde{\nu}, 0)\, e^{-\frac{(\ln[\nu/\tilde{\nu}]-y_e)^2}{4\, y_e}}\, \frac{d\tilde{\nu}}{\tilde{\nu}}. \quad (16)$$

Not including the term $2\frac{\Delta n}{x_e}$ in Eq. (15) simply leads to the replacement $y_e \rightarrow 3y_e$ in the numerator of the exponential term in Eq. (16), and after absorbing the factor $\tilde{\nu}^3/\nu^3 = \exp(-3\ln[\nu/\tilde{\nu}])$, one recovers the solution of Zeldovich & Sunyaev (1969) in the form of Eq. (2).

If we assume $\Delta n(\nu, 0) = A\, \delta(\nu - \nu_0)/\nu^2$ for the initial distortion, then from Eq. (16) we obtain

$$\Delta n(\nu, y_e) = \frac{A}{\sqrt{4\pi\, y_e}\, \nu^3} \times e^{-\frac{(\ln[\nu/\nu_0]-y_e)^2}{4\, y_e}}. \quad (17)$$

Without the inclusion of stimulated scattering in the ambient blackbody field (replacement $y_e \rightarrow 3y_e$ in Eq. (16) as explained above), as in the case of Zeldovich & Sunyaev (1969), one finds

$$\Delta n^{ZS}(\nu, y_e) = \frac{A}{\sqrt{4\pi\, y_e}\, \nu^3} \times e^{-\frac{(\ln[\nu/\nu_0]-3\, y_e)^2}{4\, y_e}} \quad (18a)$$

$$\equiv \frac{A}{\sqrt{4\pi\, y_e}\, \nu_0^3} \times e^{-\frac{(\ln[\nu/\nu_0]+3\, y_e)^2}{4\, y_e}}, \quad (18b)$$

Comparing Eqs. (17) and (18a), one can see that in terms of energy ($\Delta I_\nu \propto \nu^3 \Delta n$) the maximum of the distribution is moving like $\nu_{max}(y_e) = \nu_0\, e^{y_e}$, when including the effect of stimulated scattering, while it moves like $\nu_{max}^{ZS}(y_e) = \nu_0\, e^{3y_e}$ without this term. On the other hand, in terms of photon number ($\Delta N_\nu \propto \nu^2 \Delta n$), from Eqs. (17) and (18a) one finds that the maximum of the distribution is moving like $\nu_{max}(y_e) = \nu_0\, e^{-y_e}$, when including the effect of stimulated scattering, while it moves like $\nu_{max}^{ZS}(y_e) = \nu_0\, e^{y_e}$ without this term. In both cases the difference in the line position is a factor of e^{-2y_e}, a property that agrees with the more simple solution (8).

As an example, in Figs. 1 and 2 we show the time evolution of an initially narrow line for different values of the y-parameter. For the curves presented in Fig. 1, the effect of stimulated scattering in the blackbody ambient radiation field was included, while in Fig. 2 it was not. As expected, the maximum of the distribution is moving toward lower frequencies in the former case, while in the latter it moves toward higher frequencies. It is also remarkable that the analytic solution (16) in comparison with the full numerical solution of Eq. (14) works up to values of $y_e \sim 1$–5. It only breaks down when the photons reach the region $x_e \gtrsim 1$, where the recoil effect starts to become strong. However, in both cases (injection at $x_{e,0} = 10^{-2}$ and $x_{e,0} = 0.1$), only a small fraction of the initial number of photons reaches this region for $y_e \lesssim 1$.

One can also see that, no matter whether the background-induced effect is included or not, the line broadening is not changing much. It is only the position of the line center that is affected by this process, but not the Doppler broadening.

5. Discussion and conclusion

We have presented a new analytic solution of the Kompaneets equation for physical situations in which low-frequency photons are scattered by thermal electrons but within an intense ambient blackbody field that is very close to full thermodynamic equilibrium with the free electrons. Under these circumstances, the blackbody-induced stimulated scattering of photons slows down the motion of photons toward higher frequencies by a factor of 3 in comparison with the solution that only takes Doppler broadening and boosting into account (Zeldovich & Sunyaev 1969).

These physical conditions are met during the epoch of cosmological recombination, in particular for He III → He II-recombination, where electron scattering can still affect the cosmological recombination spectrum notably (Dubrovich & Stolyarov 1997; Rubiño-Martín et al. 2008).

The CMB blackbody spectrum is an exact solution of the Kompaneets equation when $T_e = T_\gamma$. Nevertheless, the weak low-frequency features in the presence of CMB with $T_e = T_\gamma$ are evolving and increasing their central frequencies due to the simultaneous action of the induced scattering and Doppler effect resulting after each scattering. The additional line shift towards low frequencies due to recoil effect is small because $x_e \ll 1$.

At $z \sim 6000$, i.e. where most of the He II-photons are released, one finds $y_e \sim 2 \times 10^{-3}$, so that, within the standard recombination epoch, this process is only important for *very precise* predictions of the cosmological recombination spectrum at low frequencies. Previously this effect was neglected, but its inclusion may become necessary for the purpose of using the cosmological recombination spectrum for measurements of cosmological parameters, such as the CMB temperature monopole, the specific entropy of the Universe, or the pre-stellar abundance of helium, as discussed recently (Sunyaev & Chluba 2008). Our analysis shows that the low-frequency spectral features from He III → He II recombination should then be shifted by $\Delta\nu/\nu \sim 0.2\%$ instead of $\sim 0.6\%$. At the same time, the Doppler broadening of these lines reaches ~15% at FWHM, and the typical line width is $\Delta\nu/\nu \sim$ 20–30% (Rubiño-Martín et al. 2008). For the spectral features arising during He II → He I (Rubiño-Martín et al. 2008) recombination, the corresponding line shifts are much smaller, and for those from hydrogen (e.g. see Chluba & Sunyaev 2006) recombination they are certainly negligible at the ~0.1% level.

However, if photons caused by uncompensated atomic transitions in hydrogen and helium were released much earlier ($z \gtrsim 10^4$–10^5), then the blackbody-induced stimulated scattering becomes more significant. As mentioned in the introduction, this

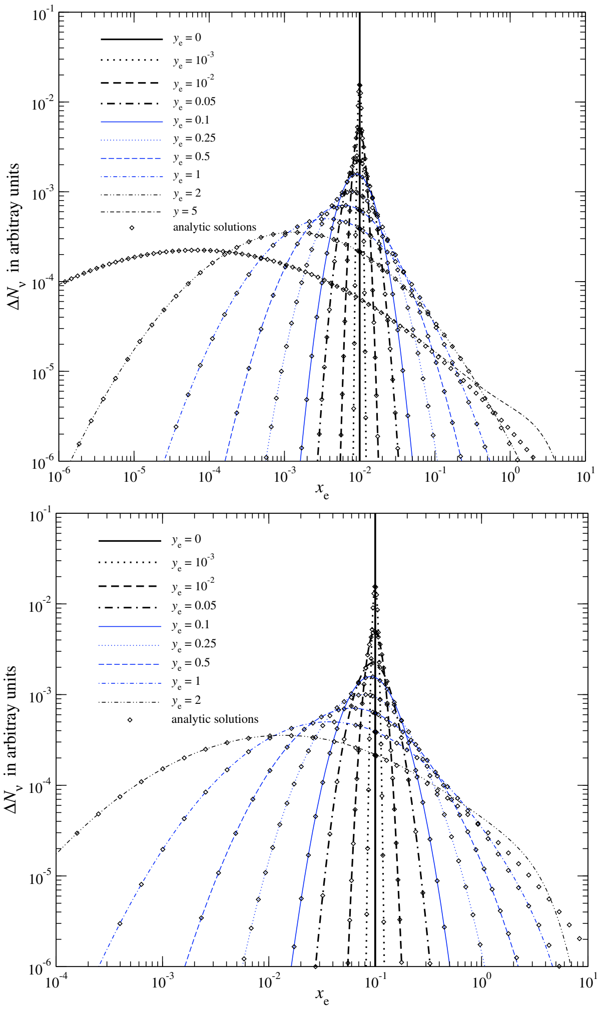

Fig. 1. Time evolution of $\Delta N_\nu = \nu^2 \Delta n$ for different values of the y-parameter *including* the effect of stimulated scattering in the blackbody ambient radiation field. The upper panel shows the case, for an initially narrow line that was injected at frequency $x_{e,0} = 10^{-2}$, while the lower panel shows the solution for injection at $x_{e,0} = 10^{-1}$. In both figures we show the results obtained by numerically solving Eq. (14) for $T_\gamma = T_e$. In addition we give the analytic solutions of the linearized problem, Eq. (15), according to Eq. (16).

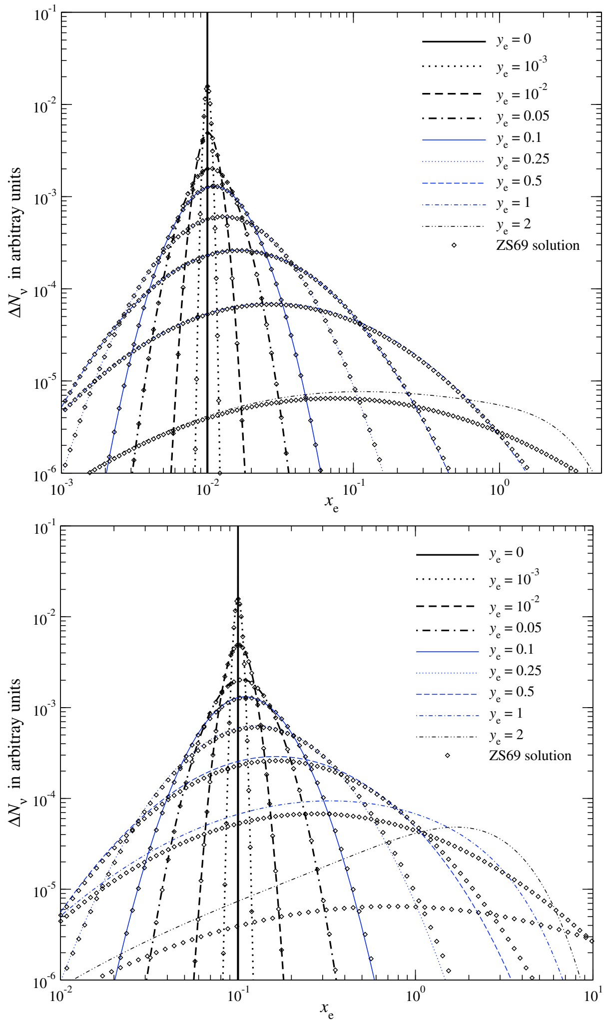

Fig. 2. Time evolution of $\Delta N_\nu = \nu^2 \Delta n$ for different values of the y-parameter *not including* the effect of stimulated scattering in the blackbody ambient radiation field ($n_{\rm pl} = 0$). The upper panel shows the case for an initially narrow line that was injected at $x_{e,0} = 10^{-2}$, while the lower panel shows the solution for injection at $x_{e,0} = 10^{-1}$. In both figures we show the results obtained by numerically solving Eq. (14) for $T_\gamma = T_e$. In addition we give the analytic solutions according to Zeldovich & Sunyaev (1969), Eq. (2).

emission can occur in connection with possible intrinsic spectral distortions of the CMB, after energy release in the early Universe (Chluba & Sunyaev 2008). Again this process has not yet been included in the computations, and will be considered in more detail in a subsequent paper.

One should stress that the discussed behavior of spectral features is only characteristic of the low-frequency part of CMB spectrum $h\nu \ll kT_\gamma$. In the dm and cm spectral ranges, this condition is fulfilled with very good accuracy, and the occupation number of photons exceeds $\sim$10. The broadening and shifting of photons in the Wien-part of the CMB spectrum shows a completely different behavior and will be discussed in a separate paper.

Finally, we also gave the solution (9) that describes the evolution of narrow spectral features in the presence of a very bright, broad-band radiation background with $n_{\rm back} \gg 1$ with power law spectral dependence. It might have interesting applications for radio lines arising inside of extra-galactic radio sources, where according to our estimates (see Sect. 3), the background-induced line shifts could be amplified and even exceed the line broadening due to the Doppler effect. A detailed analysis of such an application is beyond the scope of this paper.

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