

Secular spin-axis dynamics of exoplanets

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ABSTRACT

Context. Seasonal variations and climate stability of a planet are very sensitive to the planet obliquity and its evolution. This is of particular interest for the emergence and sustainability of land-based life, but orbital and rotational parameters of exoplanets are still poorly constrained. Numerical explorations usually realised in this situation are therefore in heavy contrast with the uncertain nature of the available data.

Aims. We aim to provide an analytical formulation of the long-term spin-axis dynamics of exoplanets, linking it directly to physical and dynamical parameters, but still giving precise quantitative results if the parameters are well known. Together with bounds for the poorly constrained parameters of exoplanets, this analysis is designed to enable a quick and straightforward exploration of the spin-axis dynamics.

Methods. The long-term orbital solution is decomposed into quasi-periodic series and the spin-axis Hamiltonian is expanded in powers of eccentricity and inclination. Chaotic zones are measured by the resonance overlap criterion. Bounds for the poorly known parameters of exoplanets are obtained from physical grounds (rotational breakup) and dynamical considerations (equipartition of the angular momentum deficit).

Results. This method gives accurate results when the orbital evolution is well known. The detailed structure of the chaotic zones for the solar system planets can be retrieved from simple analytical formulas. For less-constrained planetary systems, the maximal extent of the chaotic regions can be computed, requiring only the mass, the semi-major axis, and the eccentricity of the planets present in the system. Additionally, some estimated bounds of the precession constant allow to classify which observed exoplanets are necessarily out of major spin-orbit secular resonances (unless the precession rate is affected by the presence of massive satellites).

Key words. planets and satellites: dynamical evolution and stability – celestial mechanics – planets and satellites: general

1. Introduction

From the works by Laskar & Robutel (1993) and Laskar et al. (1993a), we know that the long-term dynamics of the terrestrial planets of the solar system feature wide chaotic regions allowing large variations of their obliquity. In particular, Mars is currently in a chaotic region extending from 0° to 60° obliquity, whereas the Earth is located in a stable region thanks to the presence of the Moon, resulting in obliquity variations of only a few degrees. Subsequent studies detailed both the past and future spin-axis evolution of the Earth (Néron de Surgy & Laskar 1997; Laskar et al. 2004a; Li & Batygin 2014a), Venus (Correia et al. 2003; Correia & Laskar 2003), and Mars (Laskar et al. 2004b). On the other hand, paleorecords for Earth show that even the very slight variations of its orbit and obliquity led to major climate changes (e.g. Weertman 1976; Hays et al. 1976). This implies that life on Earth would be very different to what it is now if the Earth had evolved in a large chaotic zone, as it would have without the stabilising effect of the Moon. This conclusion is reinforced by the fact that high-obliquity planets undergo severe seasonal variations (e.g. Spiegel et al. 2009) even for a stable obliquity. Nevertheless, conditions suitable for the very emergence of life could still be achieved: in some extreme cases, the amount of liquid water on the surface may even be favoured by large obliquity variations, as reported by Armstrong et al. (2014).

As the formation of the Moon is thought to have resulted from an accidental collision event (e.g. Hartmann & Davis 1975; Canup & Asphaug 2001; Lock et al. 2018), a “moonless” Earth

was a possible (or even likely) outcome of the planetary formation process. In the broader context of exoplanets, the dynamics of such a moonless Earth is therefore of more than simple academic interest. This motivated further studies of the structure of the chaotic region (Li & Batygin 2014b) and some additional numerical experiments (Lissauer et al. 2012).

When it comes to exoplanets, we must face the problem of the incomplete and imprecise nature of both dynamical and physical data. Except from very favourable cases, like a fast precession and an important flattening inducing detectable transit depth modulations (Carter & Winn 2010; Correia 2014), the spin orientation and the flattening of exoplanets are far from being attainable by observations. They must therefore be taken as completely free parameters, in the spirit of the work by Laskar & Robutel (1993) for the solar system planets. Moreover, the orbital properties of exoplanets are not well known either, especially the respective orientations of the orbits (including the mutual inclinations, which play a crucial role in the spin-axis dynamics). Several authors have already tackled this problem (e.g. Brassier et al. 2014; Deitrick et al. 2018; Shan & Li 2018): they used numerical integrations of the planetary system in order to build the time-dependent perturbation of the spin axis. Applied to extrasolar planets, this method requires a nominal value to be chosen for both the rotational parameters and the unknown orbital elements. The parameter space to be explored is thus very wide, meaning that even elaborate numerical explorations require some degree of arbitrariness. Consequently, the use of numerical integrations at this stage could appear to be in

contradiction with the very incomplete nature of the data. However, the secular problem (averaged over rotational and orbital motions) is not as complex as it could appear. Some suggestions for a possible analytical treatment were actually given by [Laskar \(1996\)](#) and partially exploited by [Atobe et al. \(2004\)](#), [Li & Batygin \(2014b\)](#), and [Shan & Li \(2018\)](#). In the exoplanetary case, a refined analytical theory would be very convenient, since it would give in a direct way the sensitivity of the spin-axis dynamics to the various known and unknown parameters, instead of giving a list of possible outcomes. Associated with some bounds for the unknown parameters, such a theory would give a clear range for these possible outcomes.

This was the approach used by [Atobe et al. \(2004\)](#), with the aim of finding the probability of small obliquity variations for hypothetical terrestrial planets in the habitable zone of known exoplanetary systems. Their analytical developments, though, were limited to the lowest-order approximations (their parameter space was indeed enormous since, in this case, the planet itself was hypothetical). The recent work by [Shan & Li \(2018\)](#) also contains an analytical part applied to the particular case of exoplanets Kepler-62f and Kepler-186f. This time, their calculations were mostly designed to support and explain numerical results, and they did not try to substantially improve the theory by [Atobe et al. \(2004\)](#).

In this context, the goal of this article is twofold: (i) we aim to provide a general analytical formalism for studying the long-term spin-axis dynamics of (exo)planets and (ii) to clarify what kind of information about the spin can be obtained from typical observed exoplanetary systems, that is, with numerous unknown physical and dynamical parameters. In particular, it is crucial for future studies to have a simple way to classify the observed exoplanets according to the characteristics of their spin dynamics. This would allow for the determination of the exoplanets that merit further, detailed study (in particular if a complex chaotic spin dynamics is expected) and those that have a necessarily very simple spin dynamics, thus negating the need for further numerical or analytical study. Such a general analysis will give both a qualitative view of the system if it is poorly known (in the continuation of [Atobe et al. 2004](#)), and a quantitative description of the dynamics if it is well known (as an analytical counterpart of [Laskar & Robutel 1993](#)).

This article is organised as follows: Sect. 2 recalls the Hamiltonian of the secular spin-axis dynamics and shows how it can be expanded in terms of the orbital motion parameters. The secular resonances at all orders can then be isolated and used to delimit the chaotic regions. Subsequently, Sect. 3 shows how an incomplete set of orbital elements can still be used to constrain the orbital solution of an exoplanet. Combined with the rotational breakup limit, this method allows for a preliminary classification of the “non-resonant” exoplanets, for which no chaos can appear and for which the obliquity variations are constrained by an analytical bound.

2. Analytical model of the long-term spin dynamics

2.1. Development of the Hamiltonian

We consider a system composed of a star and several planets. We study the rotational dynamics of one planet among them. For now, we consider that this planet is far from any spin-orbit resonance. Considering only the lowest-order term of the torque from the star expanded in Legendre polynomials, the Hamiltonian of rotation averaged over orbital and rotational motions is given for instance by [Laskar & Robutel \(1993\)](#) and detailed by

[Néron de Surgy & Laskar \(1997\)](#). This Hamiltonian can be written

$$\mathcal{H}(X, -\psi, t) = -\frac{\alpha}{2} \frac{X^2}{(1 - e(t)^2)^{3/2}} - \sqrt{1 - X^2} (\mathcal{A}(t) \sin \psi + \mathcal{B}(t) \cos \psi) + 2XC(t), \quad (1)$$

where the conjugate coordinates are X (cosine of obliquity) and $-\psi$ (minus the precession angle). The quantity α is referred to as the precession constant (contrary to previous studies, we prefer to exclude here the eccentricity e appearing in the denominator from the definition of α). Following the derivation proposed by [Néron de Surgy & Laskar \(1997\)](#), we obtain

$$\alpha = \frac{3\mathcal{G}m_0}{2\omega a^3} \frac{2C - A - B}{2C}, \quad (2)$$

where \mathcal{G} is the gravitational constant, m_0 is the mass of the star, a is the semi-major axis of the planet in orbit around the star, ω is its spin angular velocity, and $A \leq B \leq C$ are its momenta of inertia. The Hamiltonian (Eq. (1)) depends explicitly on time t through the eccentricity e and the functions

$$\begin{cases} \mathcal{A}(t) = \frac{2(\dot{q} + pC(t))}{\sqrt{1 - p^2 - q^2}}, \\ \mathcal{B}(t) = \frac{2(\dot{p} - qC(t))}{\sqrt{1 - p^2 - q^2}}, \end{cases} \quad C(t) = q\dot{p} - p\dot{q}, \quad (3)$$

in which $q = \sin(I/2) \cos \Omega$ and $p = \sin(I/2) \sin \Omega$, where I and Ω are the orbital inclination and the longitude of ascending node of the planet, respectively. In the following, we write $\eta \equiv \sin(I/2)$. One should note that if there is only one planet in the system (two-body problem), the obliquity is constant and the precession angle circulates with constant angular velocity $\alpha X / (1 - e^2)^{3/2}$.

We consider a scenario where the eccentricity and the inclination of the planet are small, such that we are allowed to develop the Hamiltonian in series of e and η . In the following, we present the terms up to order 3, but the method presented here can be generalised to any order (as we see below, the third order is the first one at which the eccentricity begins to play a substantial role). Using the fact that $C = \eta^2 \Omega = O(\eta^2)$, we obtain

$$\begin{aligned} \mathcal{A} &= (2 + p^2 + q^2)\dot{q} + 2pC + O(\eta^4), \\ \mathcal{B} &= (2 + p^2 + q^2)\dot{p} - 2qC + O(\eta^4). \end{aligned} \quad (4)$$

We now consider the secular orbital dynamics of the planet, resulting from the perturbations by the other planets, to be quasi-periodic. This amounts to considering that the chaos present in the orbital secular system acts on a much larger timescale than the spin dynamics under study. As we see below, this holds very well for the solar system (this methodology was first proposed by [Laskar 1996](#); it is used e.g. by [Li & Batygin 2014b](#)). In this case, we can write

$$\begin{aligned} e \exp(i\varpi) &= \sum_{j=1}^N E_j \exp(i\theta_j), \\ \eta \exp(i\Omega) &= \sum_{j=1}^M S_j \exp(i\phi_j), \end{aligned} \quad (5)$$

where ϖ is the longitude of pericentre of the planet in orbit around the star. The angles θ_j and ϕ_j evolve linearly with frequencies μ_j and ν_j , that is,

$$\theta_j(t) = \mu_j t + \theta_j^{(0)} \quad \text{and} \quad \phi_j(t) = \nu_j t + \phi_j^{(0)}, \quad (6)$$

whereas the amplitudes E_j and S_j are real constants of order e and η or smaller. Such series can be obtained either from analytical theories or from frequency analysis of numerical solutions (Laskar 1988, 1990). In a general integrable case, μ_j and ν_j are integer combinations of the fundamental frequencies of the orbital dynamics (usually noted g_k and s_k), and the series contain an infinite number of terms. Arranging the terms by decreasing amplitude, we consider here a truncation with N terms for the eccentricity and M terms for the inclination. We thus obtain

$$e^2 = \sum_{j=1}^N E_j^2 + 2 \sum_{j<k}^N E_j E_k \cos(\theta_j - \theta_k), \quad (7)$$

$$C = \sum_{j=1}^M \nu_j S_j^2 + \sum_{j<k}^M (\nu_j + \nu_k) S_j S_k \cos(\phi_j - \phi_k), \quad (8)$$

and from Eq. (4),

$$\begin{aligned} \mathcal{A} \sin \psi + \mathcal{B} \cos \psi &= 2 \sum_{j=1}^M \nu_j S_j \cos(\phi_j + \psi) \\ &+ \sum_{j=1}^M \left[\nu_j S_j^3 - 2S_j \left(\sum_{k=1}^M \nu_k S_k^2 \right) \right] \cos(\phi_j + \psi) \\ &- \sum_{j=1}^M \sum_{\substack{k=1 \\ k \neq j}}^M \nu_k S_j^2 S_k \cos(2\phi_j - \phi_k + \psi) \\ &- 2 \sum_{i=1}^M \sum_{\substack{j<k \\ j,k \neq i}}^M \nu_i S_i S_j S_k \cos(-\phi_i + \phi_j + \phi_k + \psi) \\ &+ O(\eta^4). \end{aligned} \quad (9)$$

In order to obtain an autonomous Hamiltonian, we introduce the momenta Θ_j and Φ_j conjugate to θ_j and ϕ_j . The system has now $N + M + 1$ degrees of freedom. The new Hamiltonian (that we still denote \mathcal{H}) can be written

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \varepsilon^2 \mathcal{H}_2 + \varepsilon^3 \mathcal{H}_3 + O(\varepsilon^4), \quad (10)$$

where we suppose that $O(e) = O(\eta) = O(\varepsilon)$. The different parts are, respectively,

$$\mathcal{H}_0(X, \Theta, \Phi) = -\frac{\alpha}{2} X^2 + \sum_{j=1}^N \mu_j \Theta_j + \sum_{j=1}^M \nu_j \Phi_j, \quad (11)$$

$$\varepsilon \mathcal{H}_1(X, -\psi, \phi) = -2\sqrt{1-X^2} \sum_{j=1}^M \nu_j S_j \cos(\phi_j + \psi), \quad (12)$$

$$\begin{aligned} \varepsilon^2 \mathcal{H}_2(X, \theta, \phi) &= -\frac{3\alpha}{4} X^2 \sum_{j=1}^N E_j^2 + 2X \sum_{j=1}^M \nu_j S_j^2 \\ &- \frac{3\alpha}{2} X^2 \sum_{j<k}^N E_j E_k \cos(\theta_j - \theta_k) \\ &+ 2X \sum_{j<k}^M (\nu_j + \nu_k) S_j S_k \cos(\phi_j - \phi_k), \end{aligned} \quad (13)$$

and

$$\begin{aligned} \varepsilon^3 \mathcal{H}_3(X, -\psi, \phi) &= \\ &- \sqrt{1-X^2} \sum_{j=1}^M \left[\nu_j S_j^3 - 2S_j \left(\sum_{k=1}^M \nu_k S_k^2 \right) \right] \cos(\phi_j + \psi) \\ &+ \sqrt{1-X^2} \sum_{j=1}^M \sum_{\substack{k=1 \\ k \neq j}}^M \nu_k S_j^2 S_k \cos(2\phi_j - \phi_k + \psi) \\ &+ 2\sqrt{1-X^2} \sum_{i=1}^M \sum_{\substack{j<k \\ j,k \neq i}}^M \nu_i S_i S_j S_k \cos(-\phi_i + \phi_j + \phi_k + \psi). \end{aligned} \quad (14)$$

We note that there can be no resonance among the angles ϕ_j and θ_j because they come from the quasi-periodic solution of the orbital dynamics. By definition, they are thus already “integrated”.

2.2. One perturbing term: Colombo’s top

From Eqs. (11)–(12), we conclude that at lowest-order to the perturbation, resonant angles can only be of the form $\sigma = \phi_j + \psi$. We now consider a single resonance with the term $j = p$. We introduce the resonant canonical coordinates by the linear transformation

$$\begin{cases} \sigma = \psi + \phi_p, \\ \xi = -\phi_p, \end{cases} \quad \text{and} \quad \begin{cases} \Sigma = -X, \\ \Xi = -X - \Phi_p. \end{cases} \quad (15)$$

Assuming that the system is far from any other resonance, the long-term dynamics at first order to the perturbation is given by averaging the Hamiltonian over all angles but σ . Dropping the constant terms, we get

$$\mathcal{F}(\Sigma, \sigma) = -\frac{1}{2} a \alpha \Sigma^2 + b \Sigma + c \sqrt{1-\Sigma^2} \cos \sigma, \quad (16)$$

with

$$\begin{aligned} a &= 1 + \frac{3}{2} \sum_{j=1}^N E_j^2, \\ b &= \nu_p - 2 \sum_{j=1}^M \nu_j S_j^2, \end{aligned} \quad (17)$$

$$c = -2\nu_p S_p - \nu_p S_p^3 + 2S_p \sum_{j=1}^M \nu_j S_j^2.$$

As shown in Appendix A, the Hamiltonian has the same form in the case of a 1:1 spin-orbit resonance, with a slightly different expression of the coefficients however. This would only slightly shift the position of the secular resonances considered here. In this case, the tidal damping (responsible for this capture in spin-orbit resonance) is supposed to act on a much larger timescale than the spin dynamics studied here, such that our approach still holds. Finally, applying the modified time $d\tau = \alpha dt$ to Eq. (16), we obtain the following Hamiltonian (that we still denote \mathcal{F}):

$$\mathcal{F}(\Sigma, \sigma) = -\frac{1}{2} \Sigma^2 + \gamma \Sigma + \beta \sqrt{1-\Sigma^2} \cos \sigma, \quad (18)$$

where

$$\gamma = \frac{b}{\alpha} \quad \text{and} \quad \beta = \frac{c}{\alpha}. \quad (19)$$

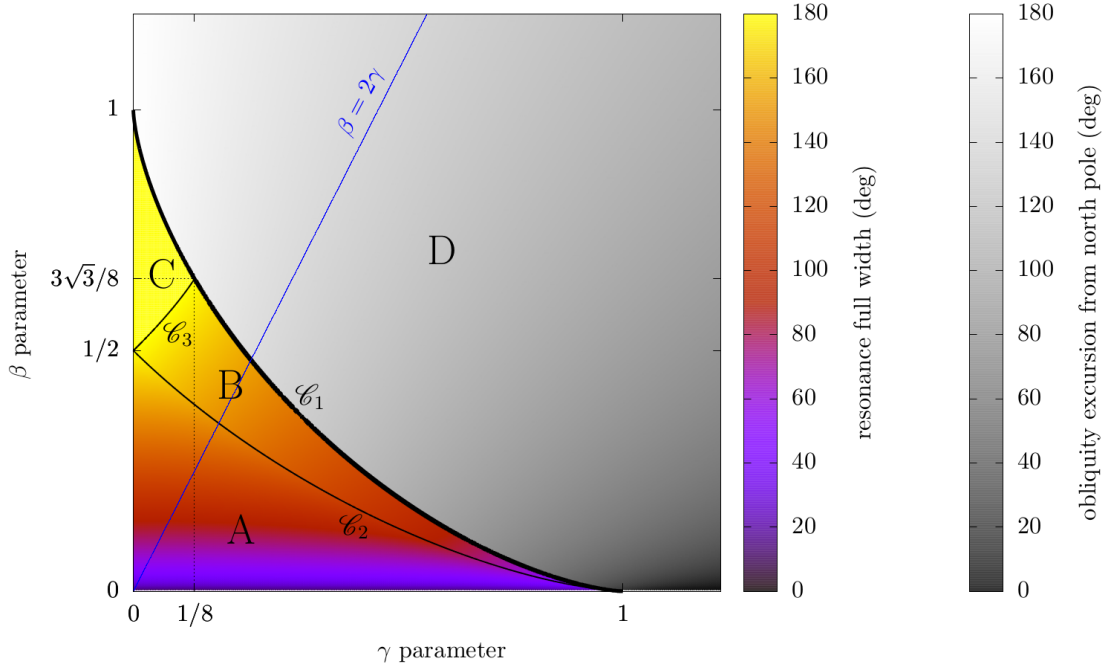


Fig. 1. Parameter space of Colombo’s top Hamiltonian (Eq. (18)). There is no separatrix (and thus no resonance) in region D, delimited by the curve \mathcal{C}_1 (Eq. (20), thick black line). The curves \mathcal{C}_2 and \mathcal{C}_3 (Eq. (25), thin black lines) delimit regions A–B and B–C, respectively. In region A, the resonance island lies between Σ_- and Σ_+ (Eq. (24)); in region B, it lies between Σ_- and $+1$; in region C, it lies between -1 and $+1$ (hence the 180° full width given by the colour scale). The problem becomes unphysical above $\beta = 2\gamma$ (blue line) since it corresponds to an amplitude $S_p > 1$ in the inclination series.

The dynamical system with Hamiltonian function (Eq. (18)) is well known. It was thoroughly studied by [Henrard & Murigande \(1987\)](#), who called it “Colombo’s top” in memory of [Colombo \(1966\)](#). In the following, we detail its characteristics of interest here, in terms of the two constant parameters γ and β .

First of all, it is enough to study the case $\gamma \geq 0$ and $\beta \geq 0$ since we get the negative cases by the transformations $\Sigma \rightarrow -\Sigma$ and $\sigma \rightarrow \sigma + \pi$, respectively. When studying the equilibrium points of the system ([Henrard & Murigande 1987](#), or Appendices B.1 and B.2), we find that the phase space can have two different geometries according to the value of γ and β (see Fig. 2 for the phase portraits). The boundary between these two regions of the parameter space is the curve

$$\mathcal{C}_1 = \{ \gamma, \beta \geq 0 : \gamma^{2/3} + \beta^{2/3} = 1 \}. \quad (20)$$

Below \mathcal{C}_1 (regions A, B, C of Fig. 1), the dynamical system (Eq. (18)) has four equilibrium points, which we denote (a, b, c, d) . The equilibrium points c and d merge along the curve \mathcal{C}_1 , and disappear above it (region D of Fig. 1). Their respective positions are

$$\begin{cases} \Sigma_a \in [0, +1] , \sigma_a = 0 \rightarrow \text{elliptic (A,B,C,D)}, \\ \Sigma_b \in [-1, 0] , \sigma_b = \pi \rightarrow \text{elliptic (A,B,C,D)}, \\ \Sigma_c \in [0, +1] , \sigma_c = \pi \rightarrow \text{hyperbolic (A,B,C)}, \\ \Sigma_d \in [0, +1] , \sigma_d = \pi \rightarrow \text{elliptic (A,B,C)}. \end{cases} \quad (21)$$

The values $\Sigma_{a,b,c,d}$ have explicit expressions in terms of γ and β , as given in Appendix B.1 (they correspond to the different roots of a quartic equation). Since the resonant angle $\psi + \phi_p$ is equal to 0 or π for each of these equilibrium points, they all correspond to configurations where the spin axis, the normal to the orbit (reduced to its p th harmonic), and the normal to the reference plane are all in the same plane. As such, they are

commonly called “Cassini’s states” and labelled (1, 2, 3, 4) after [Peale \(1969\)](#), corresponding to the equilibrium points (c, a, b, d) .

We note the limiting cases

$$\text{for } \beta \leq 1 : \begin{cases} \lim_{\gamma \rightarrow 0} \Sigma_a = \lim_{\gamma \rightarrow 0} \Sigma_c = 0 \\ \lim_{\gamma \rightarrow 0} \Sigma_d = -\lim_{\gamma \rightarrow 0} \Sigma_b = \sqrt{1 - \beta^2} \end{cases} \quad (22)$$

$$\text{for } \beta \geq 1 : \lim_{\gamma \rightarrow 0} \Sigma_a = \lim_{\gamma \rightarrow 0} \Sigma_b = 0,$$

and

$$\text{for } \gamma \leq 1 : \begin{cases} \lim_{\beta \rightarrow 0} \Sigma_a = \lim_{\beta \rightarrow 0} \Sigma_c = \gamma \\ \lim_{\beta \rightarrow 0} \Sigma_d = -\lim_{\beta \rightarrow 0} \Sigma_b = 1 \end{cases} \quad (23)$$

$$\text{for } \gamma \geq 1; \lim_{\beta \rightarrow 0} \Sigma_a = -\lim_{\beta \rightarrow 0} \Sigma_b = 1.$$

We are now interested in the width of the resonance, that is, the interval of Σ enclosed in the separatrix emerging from the hyperbolic fixed point c and containing the fixed point a . A pendulum approximation can be obtained for small values of β (as used by [Atobe et al. 2004](#) or [Li & Batygin 2014b](#)), but this approximation is no longer valid when β grows. Since an analytical expression can be derived even in the general case, we use it here. The computations (Appendix B.5) lead to the following extreme values of Σ spanned by the resonance:

$$\Sigma_{\pm} = 2\gamma - \Sigma_c \pm 2 \sqrt{-\beta^2 + \beta \sqrt{1 - \Sigma_c^2}}. \quad (24)$$

They are defined whenever Σ_c itself is defined (regions A, B, C of Fig. 1). At this point, it is important to note that the coordinates (Σ, σ) are singular at $\Sigma = \pm 1$ since the problem actually

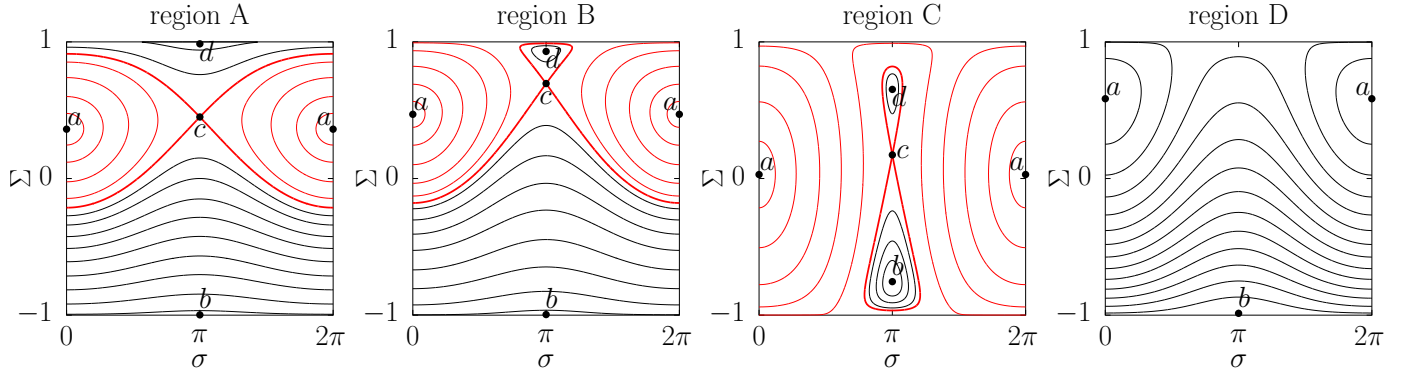


Fig. 2. Examples of phase portraits for the four regions of Fig. 1. The equilibrium points are labelled as in Eq. (21). The level curves of the Hamiltonian are drawn with black lines out of the resonance island and with red lines inside the resonance. The parameters chosen are, from A to D: $(\gamma, \beta) = (0.4, 0.1)$; $(0.55, 0.15)$; $(0.05, 0.7)$; and $(0.8, 0.3)$.

takes place on the sphere (Henrard & Murigande 1987). We must therefore carefully study the meaning of the limits (Eq. (24)) when one of them crosses ± 1 . This leads to two other limits in the parameter space, as the curves

$$\begin{aligned} \mathcal{C}_2 &= \{ \gamma, \beta \geq 0 : 8\beta^2 = 1 - 20\gamma - 8\gamma^2 + (1 + 8\gamma)^{3/2} \}, \\ \mathcal{C}_3 &= \{ 0 \leq \gamma \leq 1/8, \beta \geq 0 : \\ &\quad 8\beta^2 = 1 + 20\gamma - 8\gamma^2 + (1 - 8\gamma)^{3/2} \} \end{aligned} \quad (25)$$

(see Appendices B.3 and B.4), delimiting the regions A–B and B–C of Fig. 1, respectively. We note that \mathcal{C}_1 and \mathcal{C}_2 intersect at $(\gamma; \beta) = (1; 0)$, \mathcal{C}_1 and \mathcal{C}_3 intersect at $(1/8; 3\sqrt{3}/8)$, and \mathcal{C}_2 and \mathcal{C}_3 intersect at $(0; 1/2)$. Contrary to \mathcal{C}_1 , the boundaries \mathcal{C}_2 and \mathcal{C}_3 do not correspond to actual bifurcations of the dynamical system, but only to the limits where the resonant island contains the north pole of the sphere (region B) and both poles of the sphere (region C). Hence, the resonance lies in $[\Sigma_-; \Sigma_+]$ in region A; in $[\Sigma_-; +1]$ in region B; and in $[-1; +1]$ in region C (see Fig. 2). In region D, there is no more resonance (the separatrix disappears), but the obliquity can still vary substantially. In Fig. 1, the colour shades in region D show the oscillation amplitude of the obliquity as the trajectory passes through $\Sigma = 1$.

When $\beta \rightarrow 0$, the width of the island tends to 0 and all the level curves in the (σ, Σ) plane tend to be horizontal. In this limit, the resonance width is small and almost independent of γ (pendulum approximation). When $\gamma \rightarrow 0$, the phase portrait in the (σ, Σ) plane tends to be symmetric with respect to the line $\Sigma = 0$.

From Eq. (19), we note that

$$\gamma = \frac{\nu_p}{\alpha} + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \beta = -2S_p\gamma + \mathcal{O}(\varepsilon^2). \quad (26)$$

Remembering that S_p is the amplitude of a given term in the inclination series (Eq. (5)), this means that the parameter region above the line $\beta = 2\gamma$ in Fig. 1 (that is, $|S_p| = 1$) cannot be reached in this problem.

As a simple rule of thumb, one can consider that β controls the resonance width, and that γ controls the location of the resonance centre (see the Hamiltonian in Eq. (18)). From Eq. (26), we know that β is proportional to the amplitude S_p . Therefore, the resonance widths are larger in hot planetary systems, for which the mutual inclinations are large. This was exploited by Boué & Laskar (2010) in their scenario for tilting the spin-axis of Uranus. On the contrary, the secular system cannot produce any obliquity variation if the mutual inclinations are exactly zero. One

must keep in mind that β depends on the precession constant α (Eq. (2)) as well, meaning that “small” mutual inclinations do not guarantee that the resonances are thin. For the terrestrial planets of the solar system, some values of α produce first-order resonances larger than 70° , even if the mutual orbital inclinations are modest (see Sect. 2.3).

Contrary to β , the magnitude of γ cannot be easily traced back from the planetary architecture: the first-order secular spin-orbit resonances of any planet can be located anywhere between 0° and 180° of obliquity. The large majority of them actually lie in $[0^\circ; 90^\circ]$ because most of the frequencies ν_p are negative (the explanation for this property is given in Sect. 3.1).

2.3. Overlap of first-order resonances

Going back to the full Hamiltonian (Eq. (10)), the main chaotic regions of the system can be estimated as the overlap of the first-order resonances taken separately (Chirikov’s criterion). With the analytical expression of their respective widths given in Sect. 2.2, the computation of the overlapping regions is straightforward for any quasi-periodic representation (Eq. (5)) of the orbital motion.

When comparing two secular terms, we note that no chaotic zone can form if at least one of them is in region D, because there is no separatrix in D. This is well verified by Poincaré sections. A direct consequence of this property is that if all the terms of the quasi-periodic series are in region D, the secular dynamics of the obliquity cannot be chaotic at first order. Moreover, if all the S_i coefficients are small (low inclination regime), so are the corresponding β coefficients, resulting in virtually no secular variation of the obliquity (see the shades of grey in Fig. 1). A no-chaos criterion can be obtained even if the amplitudes S_i are not known: we simply have to verify that $\gamma_i \approx \nu_i/\alpha > 1 \forall i = 1, \dots, M$. On the contrary, if $\gamma_i < 1$ for at least two i , the existence of a chaotic region is possible, but not guaranteed. These properties are discussed further in Sect. 3.4.

This method can be easily checked in the case of the solar system, since the secular spin dynamics of all planets have been studied in the literature. Moreover, a very accurate quasi-periodic approximation of the orbital dynamics can be obtained, since the properties and initial conditions of the planets are very well known. As an example, the upper row of Fig. 3 shows the widths and overlaps of first-order resonances for the terrestrial planets (light and dark-red regions). This figure was produced by applying the previous analytical formulas to the orbital series of Laskar (1990), containing more than 50 terms in both

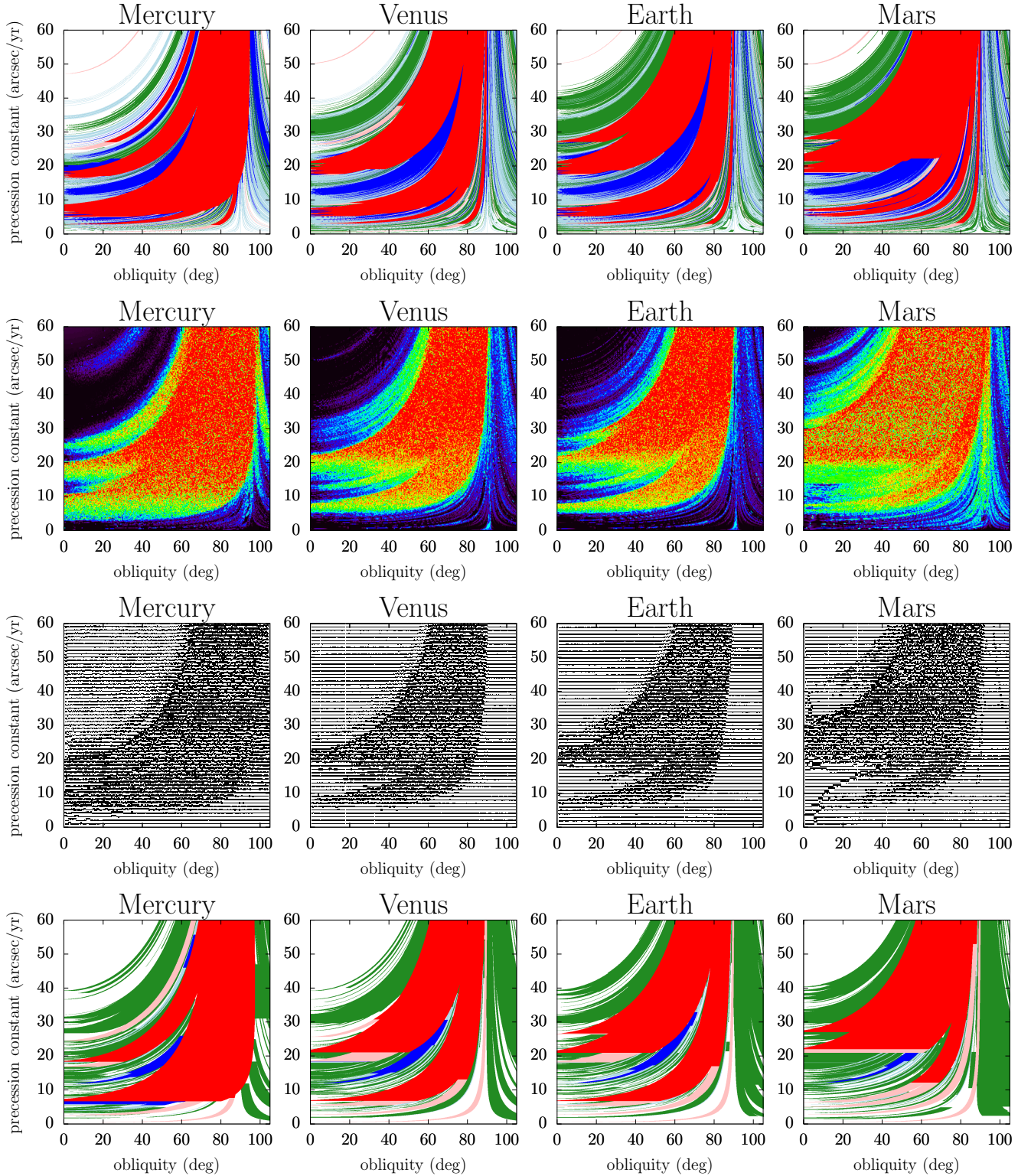


Fig. 3. *Top row:* estimate of the chaotic regions of the spin dynamics as the superposition of secular spin-orbit resonances. The orbital evolution of each planet is approximated by the synthetic representation of [Laskar \(1990\)](#), as detailed in [Appendix F](#). Light-red and dark-red regions represent the first-order resonances and their overlaps, respectively ([Colombo's top Hamiltonian, Sect. 2.2](#)); light-blue and dark-blue regions represent the second-order resonances and their overlap ([Sect. 2.4](#)); and green regions represent the overlap of third-order resonances. The non-overlapping third-order resonances are not indicated because they are very thin and thus unimportant for a global picture of the dynamics. *Second row:* as a comparison, the system given by [Eq. \(1\)](#) is integrated numerically with the same orbital model (quasi-periodic decomposition of [Laskar 1990](#)), and a frequency map analysis is performed to locate the chaotic zones. The colour scale goes from black (no chaos), to red (strong chaos). *Third row:* same maps obtained from a more detailed model in which the orbital evolution is directly taken from a numerical integration (adapted from [Laskar & Robutel 1993](#)). In the weakly chaotic zones, the dots are shifted vertically according to the level of chaos; in the strongly chaotic zones, they are plotted in boldface. *Bottom row:* same as top row, but the long-term orbital evolution of each planet is approximated by the Lagrange-Laplace system ([Sect. 3.1](#)). The eight planets of the solar system are included, with the initial conditions of [Bretagnon \(1982\)](#).

eccentricity and inclination (see Appendix F). We note that most of the first-order resonances overlap (there are almost no light-red regions). This results in wide chaotic zones even if the individual amplitudes S_i are small. However, the full extent of the chaotic regions given by the frequency analysis (Fig. 3, second and third rows) cannot be retrieved by only considering first-order resonances. The following section is therefore dedicated to second- and third-order resonances.

2.4. Higher order resonances

Outside of first-order resonances, we can use a near-identity canonical change of coordinates in order to suppress the angular dependency at first order (as already used in a similar context by Li & Batygin 2014b). Let us consider an intermediary Hamiltonian $\mathcal{X} = \varepsilon\mathcal{X}_1$, such that the current coordinates are obtained from the new ones through its flow at time 1. The Hamiltonian in the new coordinates is then

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \varepsilon\tilde{\mathcal{H}}_1 + \varepsilon^2\tilde{\mathcal{H}}_2 + \mathcal{O}(\varepsilon^3), \quad (27)$$

where

$$\begin{aligned} \tilde{\mathcal{H}}_0 &= \mathcal{H}_0, \\ \tilde{\mathcal{H}}_1 &= \mathcal{H}_1 + \{\mathcal{X}_1, \mathcal{H}_0\}, \\ \tilde{\mathcal{H}}_2 &= \mathcal{H}_2 + \{\mathcal{X}_1, \mathcal{H}_1\} + \frac{1}{2}\{\mathcal{X}_1, \{\mathcal{X}_1, \mathcal{H}_0\}\}. \end{aligned} \quad (28)$$

In these expressions, Poisson's brackets are defined as

$$\{f, g\} = \sum_j \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right), \quad (29)$$

where the pairs (p_j, q_j) are conjugate variables, p_j being the momentum and q_j the coordinate. In order to suppress the angular dependency at order 1, the Hamiltonian \mathcal{X} must fulfil the homological equation

$$\tilde{\mathcal{H}}_1 = \mathcal{H}_1 + \{\mathcal{X}_1, \mathcal{H}_0\} = \overline{\mathcal{H}}_1, \quad (30)$$

in which $\overline{\mathcal{H}}_1$ is the zeroth-order term of the multidimensional Fourier decomposition of \mathcal{H}_1 (average of \mathcal{H}_1 over all angles), which is here equal to zero. By matching the terms of the Fourier decomposition of \mathcal{H}_1 and \mathcal{X}_1 one by one, the solution to the homological equation is

$$\varepsilon\mathcal{X}_1(X, \phi, -\psi) = -2\sqrt{1-X^2} \sum_{j=1}^M \frac{\nu_j S_j}{\nu_j + \alpha X} \sin(\phi_j + \psi). \quad (31)$$

Injecting this function into the expression of the new Hamiltonian (Eq. (28)), we get $\tilde{\mathcal{H}}_1 = 0$ as required, and the second-order term $\tilde{\mathcal{H}}_2$ is given in Appendix C. The only possible resonant angles at second order have the form $\sigma = \phi_j + \phi_k + 2\psi$. The width of second- and higher order resonances is quite small, so that their separatrices can be computed assuming that X is near the exact resonance (pendulum approximation). Accordingly, the centre and half width of the second-order resonances are computed in Appendix C and given in the first line of Table 1.

Outside of both first-order and second-order resonances, the same method can be used to compute the location and width of third-order resonances. An intermediary Hamiltonian of the form $\mathcal{X} = \varepsilon\mathcal{X}_1 + \varepsilon^2\mathcal{X}_2$ is used, in which \mathcal{X}_2 must satisfy a second

homological equation (see Appendix D). The possible resonant angles, as well as the centre and half width of all the third-order resonances are given in Table 1. As before, the same method can be used in the case of synchronous rotation, by adding the term $-\alpha_r(1+X)^2/2$ to \mathcal{H}_0 (Appendix A). This would only slightly shift the resonances.

With these values, it is straightforward to compute the overlap regions of every possible second- and third-order resonance. Resonances of order 2 or 3 with a centre located inside a resonance of lower order are not considered (in other words, low-order resonances are assumed to be unaffected by higher order ones). The result is shown in the top row of Fig. 3 for the inner solar system (blue and green zones). We obtain a much better match with the frequency map analysis, showing the importance of high-order resonances in this context. Indeed, numerous frequencies of the quasi-periodic representation are quite close to each other, which implies that the corresponding resonances overlap massively. Hence, even if the second- and third-order resonances are thin, most of them are located one after another, resulting in large chaotic zones. The chaotic diffusion is however slower for higher order resonances. Moreover, in the real solar system, in which the secular orbital frequencies are actually not fixed but vary slowly (Laskar 1990), the diffusion of the obliquity will be facilitated by small modulations of the resonance locations.

We note that third-order resonances include terms mixing both eccentricity and inclination (last line of Table 1). The existence of these terms shows that the $s_6 + g_5 - g_6$ resonance, which is known to play an important role in the future obliquity evolution of the Earth (Laskar et al. 1993b, 2004a), has two different origins: one is a first-order resonance with ν_{23} , and the other is a third-order resonance between μ_1, μ_{10} and ν_{12} (the numbering refers to Laskar et al. 2004a). The amplitude of the first resonance, which is the one emphasised in the literature, is larger by a factor of thirty. This resonance is present in the synthetic representation used in this paper (see the term with frequency $-50.30212'' \text{ yr}^{-1}$ in Table F.1). In the top row of Fig. 3, it appears as a thin isolated resonance (upper-left corner of the graphs).

Finally, since tidal dissipations are much more efficient in decreasing the eccentricity than the inclination, one can imagine a planet with an initially chaotic obliquity wandering in a mixed-type-resonance overlap region, becoming frozen out of resonance when the amplitudes E_i decrease due to tidal dissipation. As shown by Laskar et al. 2012, the eccentricity amplitudes of the whole planetary system can be damped even if only one planet dissipates energy with the star. Therefore, the eccentricity modes of an external planet can be damped even if it is not itself subject to tidal dissipation.

3. Application to exoplanetary systems

In the previous sections, we saw that in the low-eccentricity and low-inclination regime, the long-term rotational dynamics of planets can be studied very efficiently by a simple analytical model. However, even if numerous exoplanetary systems are known nowadays, most of the information required to characterise the rotation of their planets remains poorly constrained. This information can be split into two groups: (i) the orbital dynamics (amplitudes and frequencies of the quasi-periodic representation) and (ii) the rotation parameters (α coefficient). In this section, we outline how these quantities can be estimated from physical and dynamical arguments, even with scarce data.

Table 1. Critical angle, location, and half width of every second- and third-order secular spin-orbit resonance.

σ	X_0	K (for a half width $2\sqrt{ K }$)
$\phi_j + \phi_k + 2\psi$	$-\frac{1}{2\alpha}(\nu_j + \nu_k)$	$8(1 - X_0^2) \frac{\nu_j \nu_k}{(\nu_j - \nu_k)^2} S_j S_k$
$2\phi_j + \phi_k + 3\psi$	$-\frac{1}{3\alpha}(2\nu_j + \nu_k)$	$-\frac{243}{2}(1 - X_0^2)^{3/2} \frac{\alpha \nu_j^2 \nu_k}{(\nu_j - \nu_k)^4} S_j^2 S_k$
$2\phi_j - \phi_k + \psi$	$-\frac{1}{\alpha}(2\nu_j - \nu_k)$	$\frac{1}{2} \sqrt{1 - X_0^2} \frac{2(2\nu_j^2 + 3\nu_j \nu_k - \nu_k^2)(\nu_j - \nu_k)^3 + \nu_j^2 \nu_k (\nu_k^2 - \alpha^2)}{\alpha(\nu_j - \nu_k)^4} S_j^2 S_k$
$-\phi_i + \phi_j + \phi_k + \psi$	$-\frac{1}{\alpha}(-\nu_i + \nu_j + \nu_k)$	$2 \sqrt{1 - X_0^2} \frac{P}{\alpha(\nu_i - \nu_j)(\nu_i - \nu_k)(2\nu_i - \nu_j - \nu_k)^2} S_i S_j S_k$
$\phi_i + \phi_j + \phi_k + 3\psi$	$-\frac{1}{3\alpha}(\nu_i + \nu_j + \nu_k)$	$-486(1 - X_0^2)^{3/2} \frac{\alpha \nu_i \nu_j \nu_k ((\nu_i - \nu_j)^2 + (\nu_j - \nu_k)^2 + (\nu_k - \nu_i)^2)}{(2\nu_i - \nu_j - \nu_k)^2 (2\nu_j - \nu_i - \nu_k)^2 (2\nu_k - \nu_i - \nu_j)^2} S_i S_j S_k$
$\phi_i + \theta_j - \theta_k + \psi$	$-\frac{1}{\alpha}(\nu_i + \mu_j - \mu_k)$	$-3X_0 \sqrt{1 - X_0^2} \frac{\nu_i}{\mu_j - \mu_k} S_i E_j E_k$

Notes. In the fourth line, the symbol P stands for: $4\nu_i^5 - 12\nu_i^4(\nu_j + \nu_k) + \nu_i^3(13\nu_j^2 + 13\nu_k^2 + 16\nu_j\nu_k) - 2\nu_i^2(3\nu_j^3 + 3\nu_k^3 + \nu_j^2\nu_k + \nu_j\nu_k^2) + \nu_i(\nu_j^4 + \nu_k^4 - 3\nu_j^3\nu_k - 3\nu_j\nu_k^3 - 8\nu_j^2\nu_k^2 - 2\alpha^2\nu_j\nu_k) + \nu_j\nu_k(\nu_j + \nu_k)^3$.

3.1. The Lagrange-Laplace system

Regarding the long-term orbital dynamics, one can use nominal orbital elements (either best-fit or assumed ones) and numerically integrate the equations of motion. The resulting solution can then be used directly (as did for instance [Brasser et al. 2014](#), and [Deitrick et al. 2018](#)), or put in the form of quasi-periodic series and used as shown above. However, this method puts a heavy contrast between the very uncertain nature of the orbital elements used and the refined numerical solution applied. In fact we see below that at this level of precision, the Lagrange-Laplace system is already a good-enough approximation of the orbital dynamics, up to moderate eccentricities and inclinations (and without strong effects coming from mean-motion resonances). It was used for the same purpose by [Atobe et al. \(2004\)](#) in the case of a massless hypothetical terrestrial planet.

The Lagrange-Laplace system is the lowest-order model of the long-term orbital dynamics: it uses a development of the Hamiltonian at second order of the eccentricities and inclinations, which is itself averaged over the fast angles (secular model at first order to the mutual perturbations). Let us write

$$z_k = e_k \exp(i\varpi_k) \quad \text{and} \quad \zeta_k = \sin \frac{I_k}{2} \exp(i\Omega_k), \quad (32)$$

where the index $k = 1, 2, \dots, N$ represents a given planet of the system. Writing \mathbf{z} and $\boldsymbol{\zeta}$ the vectors of all z_k and ζ_k , the equations of motion in the Lagrange-Laplace approximation are

$$\dot{\mathbf{z}} = iA\mathbf{z} \quad \text{and} \quad \dot{\boldsymbol{\zeta}} = iB\boldsymbol{\zeta}, \quad (33)$$

where the real matrices A and B are only functions of the masses and semi-major axes. They can be retrieved from the lowest-order terms in eccentricity and inclination of the orbital Hamiltonian. Organising the planets by increasing semi-major axes, we get from [Laskar & Robutel \(1995\)](#):

$$A_{jj} = n_j \sum_{k=1}^{j-1} \frac{m_k}{m_0} C_3 \left(\frac{a_k}{a_j} \right) + n_j \sum_{k=j+1}^N \frac{m_k}{m_0} \frac{a_j}{a_k} C_3 \left(\frac{a_j}{a_k} \right),$$

$$A_{jk} = \begin{cases} 2n_j \frac{m_k}{m_0} C_2 \left(\frac{a_k}{a_j} \right), & \text{if } k < j, \\ 2n_j \frac{m_k}{m_0} \frac{a_j}{a_k} C_2 \left(\frac{a_j}{a_k} \right), & \text{if } k > j, \end{cases} \quad (34)$$

and

$$B_{jj} = -n_j \sum_{k=1}^{j-1} \frac{m_k}{m_0} C_3 \left(\frac{a_k}{a_j} \right) - n_j \sum_{k=j+1}^N \frac{m_k}{m_0} \frac{a_j}{a_k} C_3 \left(\frac{a_j}{a_k} \right),$$

$$B_{jk} = \begin{cases} n_j \frac{m_k}{m_0} C_3 \left(\frac{a_k}{a_j} \right), & \text{if } k < j, \\ n_j \frac{m_k}{m_0} \frac{a_j}{a_k} C_3 \left(\frac{a_j}{a_k} \right), & \text{if } k > j, \end{cases} \quad (35)$$

in which $n_j^2 a_j^3 = \mathcal{G}(m_0 + m_j)$, and the functions $C_2(\alpha)$ and $C_3(\alpha)$ are expressed in terms of the Laplace coefficients $b_s^{(k)}$:

$$C_2(\alpha) = \frac{3}{8} \alpha b_{3/2}^{(0)}(\alpha) - \frac{1}{4} (1 + \alpha^2) b_{3/2}^{(1)}(\alpha),$$

$$C_3(\alpha) = \frac{1}{4} \alpha b_{3/2}^{(1)}(\alpha), \quad (36)$$

(see [Laskar & Robutel 1995](#), [Laskar et al. 2012](#) or [Murray & Dermott 1999](#)). The equations of motion for \mathbf{z} and $\boldsymbol{\zeta}$ are decoupled and linear, such that the solution can be obtained by diagonalising the matrices A and B . Its expression can be taken directly from [Laskar et al. \(2012\)](#) and has the form of quasi-periodic series (Eq. (5)) as required by our model. The frequencies g_k and s_k are the eigenvalues of A and B , and the amplitude of the term k for the planet j is

$$E_k^{(j)} = \left| P_{jk} \sum_{i=1}^N P_{ki}^{-1} z_i(0) \right| \quad (\text{eccentricity series}),$$

$$S_k^{(j)} = \left| Q_{jk} \sum_{i=1}^N Q_{ki}^{-1} \zeta_i(0) \right| \quad (\text{inclination series}), \quad (37)$$

where (P, Q) are the matrices composed of the eigenvectors of (A, B) , the matrices (P^{-1}, Q^{-1}) are their inverses, and $z_i(0), \zeta_i(0)$ are the initial conditions of planet i . In this case, we note that there is a single term for each proper frequency of the system; the frequencies μ_j and ν_j of the quasi-periodic representation are thus directly equal to one of the g_k and s_k , respectively.

From the conservation of total orbital angular momentum, one of the inclination proper frequencies s_k is identically equal to zero (matrix B has rank deficiency of order 1). The inclination series thus have $M = N - 1$ terms, whereas the eccentricity

series have N terms. Moreover, all the inclination proper frequencies are negative (this can be shown from the Geršgorin circles theorem; see Appendix E).

The result, in terms of chaotic zones for the spin dynamics, is shown in the bottom row of Fig. 3. Although the match with the numerical maps (Fig. 3, second and third rows) is not as good as when we used the synthetic representation of the orbital dynamics (Fig. 3, top row), the estimate obtained is still remarkably good considering the uncertainties surrounding the elements of an exoplanetary system¹. Using this approach with the nominal orbital elements given by Brassier et al. (2014) and Deitrick et al. (2018), we retrieve from analytical formulas their maps showing the possible obliquity variations of HD 40307 g and Kepler-62 f, in terms of the locations and widths of the secular spin-orbit resonances (see Fig. 8 by Brassier et al. 2014 and Figs. 5, 6, 10, 11 by Deitrick et al. 2018). The differences of oscillation amplitude that they observe are a natural consequence of the initial position of the planet with respect to the resonance centre (see Fig. 2). This shows that the Lagrange-Laplace system, associated with the development of the Hamiltonian (Sect. 2), is enough to obtain the level of detail required for studying the long-term rotation of exoplanets up to moderate eccentricities and inclinations. The use of a more elaborate model would add no substantial information, owing to the large uncertainties of the exoplanetary system under study.

3.2. Maximisation of E_k and S_k

Unfortunately, several orbital elements remain unknown for most of the observed exoplanetary systems. The unknown elements usually include the mutual inclinations and the relative longitudes of the ascending node. From now on, we suppose that only the masses, the semi-major axes, and the eccentricities are known for all planets of the system. In this case, the Lagrange-Laplace matrices A and B can still be computed since they only depend on the masses and the semi-major axes. We thus obtain the two sets of frequencies μ_j and ν_j . Because the initial conditions $z_i(0)$ and $\zeta_i(0)$ appearing in Eq. (37) are not fully known, the goal here is to obtain the maximum possible value of the amplitudes E_k and S_k according to the available data.

For the eccentricity, this amounts to maximising the modulus of a sum of complex numbers with unknown phase. The result is therefore simply the sum of the moduli:

$$\max [E_k^{(j)}] = |P_{jk}| \sum_{i=1}^N e_i |P_{ki}^{-1}|, \quad (38)$$

using the fact that by definition, $|z_i(0)| = e_i$. The problem is more complex for the inclinations, since both the amplitudes and the phases of the initial conditions $\zeta_i(0)$ are unknown. It is therefore necessary to introduce additional arguments, either from a physical or dynamical point of view. Guided by statistics on the orbital excitation due to close encounters, Atobe et al. (2004), while dealing mostly with systems with a single observed planet, imposed $I = e/2$ for each of them. This law is also in agreement with statistical distributions of observed exoplanetary systems (Xie et al. 2016). In our case, though, the application of this statistical result as a strict rule for each planet of a multi-planet system seems simplistic. We opt here for the hypothesis by Laskar & Petit (2017) of equipartition of the angular momentum

¹ Some subtle dynamical effects are not reproduced by the Lagrange-Laplace system, like the $s_6 + g_5 - g_6$ first-order resonance mentioned in Sect. 2.4.

deficit (AMD) among the secular degrees of freedom. As these latter authors point out, this hypothesis is motivated both by theoretical arguments on chaotic diffusion in the secular dynamics (Laskar 1994, 2008) and by the aforementioned correlations in observed distributions. As shown below, equipartition of the AMD allows one to smooth the statistical law over all the planets contained in the system. Let us introduce the ‘‘coplanar AMD’’ of a planetary system, that is, the AMD it would have if it was strictly coplanar:

$$C_p = \sum_{j=1}^N \Lambda_j \left(1 - \sqrt{1 - e_j^2}\right), \quad (39)$$

where

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j} \sqrt{\mathcal{G}(m_0 + m_j) a_j}. \quad (40)$$

Contrary to Laskar & Petit (2017), we use here the Hamiltonian decomposition of Laskar & Robutel (1995), where the integrable part is the Sun-planet two-body problem. This is also the decomposition chosen when expressing the matrices A and B of the Lagrange-Laplace system (Eqs. (34)–(35)). The AMD equipartition hypothesis amounts to considering that the total AMD of the system,

$$C = \sum_{j=1}^N \Lambda_j \left(1 - \sqrt{1 - e_j^2} \cos I_j\right) \quad (41)$$

is equal to

$$C = 2 C_p. \quad (42)$$

Hence, even if the individual orbital inclinations are not known, the so-obtained value of C gives a bound for them. For instance, the maximum possible value of the inclination of the k th planet is given by

$$\cos [\max I_k] = \max \left[1 - \frac{C_p}{\Lambda_k \sqrt{1 - e_k^2}}, -1 \right]. \quad (43)$$

In our case, we are trying to maximise the quantity

$$\max [S_k^{(j)}] = |Q_{jk}| \sum_{i=1}^N \sin \frac{I_i}{2} |Q_{ki}^{-1}|, \quad (44)$$

obtained from Eq. (37) with unknown Ω_i , using the constraint from Eq. (42). This constraint can be rewritten

$$Z = \sum_{i=1}^N c_i \eta_i^2, \quad (45)$$

where $\eta_i = \sin(I_i/2)$, $c_i = 2\Lambda_i \sqrt{1 - e_i^2}$ and $Z = C_p$, whereas the quantity to be maximised can be written

$$Y = \sum_{i=1}^N b_i \eta_i, \quad (46)$$

where $b_i = |Q_{ki}^{-1}|$. The coefficients c_i and b_i are all positive, and $0 \leq \eta_i \leq 1$. Equation (45) forms a hyper-ellipsoid, whereas the

quantity to be maximised (Eq. (46)) forms a hyperplane. Except from particular cases that we dismiss here, there is thus only one solution for the maximisation of Y , which corresponds to the tangency of the plane and the ellipsoid. This implies that the two gradients are collinear:

$$\nabla Z = \lambda \nabla Y \iff \eta_i = \lambda \frac{b_i}{2c_i} \quad \forall i = 1, \dots, N, \quad (47)$$

where $\lambda > 0$ by definition of η_i . We get the value of λ from the imposed value of Z :

$$Z = \sum_{i=1}^N \lambda^2 \frac{b_i^2}{4c_i} \iff \lambda^2 = \frac{Z}{\sum_{i=1}^N \frac{b_i^2}{4c_i}}. \quad (48)$$

The maximum of Y with the constraint Z is thus

$$\max[Y] = \sqrt{Z \sum_{i=1}^N \frac{b_i^2}{c_i}}. \quad (49)$$

Going back to the original notations, this finally gives

$$\max[S_k^{(j)}] = |Q_{jk}| \sqrt{C_p \sum_{i=1}^N \frac{(Q_{ki}^{-1})^2}{2\Lambda_i \sqrt{1 - e_i^2}}}. \quad (50)$$

However, Eq. (47) does not take into account the condition that all the η_i are smaller than 1. In practice, if the value obtained for λ implies that one or several η_i are larger than 1, we simply have to fix them to 1 and use the same resolution method iteratively² with the remaining η_i (changing the definition of Z and Y accordingly).

Table 2 shows the comparison of the amplitudes obtained for the Earth with the full Lagrange-Laplace system, and their maximisation supposing that the mutual inclinations and longitudes of the node are unknown. As shown in Fig. 4, a chaotic map can be obtained using these maximum values. However, we note that each maximisation is specific to one single amplitude since it implies a distinct set of inclination values. Therefore, taking all the maximum amplitudes at once, as if they formed one single representation, gives a large upper bound for the chaotic zones. Moreover, due to the large amplitudes of the series obtained, the small-width approximation for second- and third-order resonances does not necessarily hold.

For simple systems, such as those studied by Brassier et al. (2014), Deitrick et al. (2018), or Shan & Li (2018), the resonances are thin and well separated. A picture of the resonant regions (which do not overlap in these cases) is therefore enough to give a clear view of the dynamics. Using the maximised amplitudes gives the largest possible widths of the resonances, which, in turn shows the maximum obliquity variations and their locations. We fully retrieve their results. In contrast to these simple and ordered dynamics, Fig. 5 shows as an illustration of the maximised chaotic regions for the GJ 3293 system. The available orbital elements are taken from Astudillo-Defru et al. (2017), who pointed out that GJ 3293 d is in the habitable zone. In the spin-down process toward synchronous rotation due to tidal dissipative effects from the star (thus decreasing the precession

² From the form of the constraint (Eq. (45)), decreasing one η_i to 1 implies that at least one of the remaining η_i should increase; from the solution (Eq. (47)), this actually means that *all* the remaining η_i increase.

Table 2. Secular representation of the Earth orbital dynamics given by the Lagrange-Laplace theory.

μ_i (" yr ⁻¹)	E_i ($\times 10^5$)	$\max[E_i]$ ($\times 10^5$)	ν_i (" yr ⁻¹)	S_i ($\times 10^5$)	$\max[S_i]$ ($\times 10^5$)
3.7137	1628	1974	0.0000	1377	2420
18.0043	1492	1917	-18.7456	1222	1989
7.3460	1490	3286	-6.5701	409	3424
17.3308	1057	2381	-5.2008	425	1606
5.4615	404	600	-17.6358	226	929
22.2944	247	385	-25.7514	141	733
2.7015	61	194	-2.9039	87	993
0.6333	1	2	-0.6778	65	895

Notes. The eight planets of the Solar System are included. In the third column, each amplitude is maximised in the case where both the mutual inclinations and the longitudes are unknown, assuming the equipartition of AMD between secular degrees of freedom (since the Solar System is hierarchically AMD stable, the AMD of the inner and outer parts were taken separately, see Laskar & Petit 2017). The initial conditions and physical parameters are taken from Bretagnon (1982).

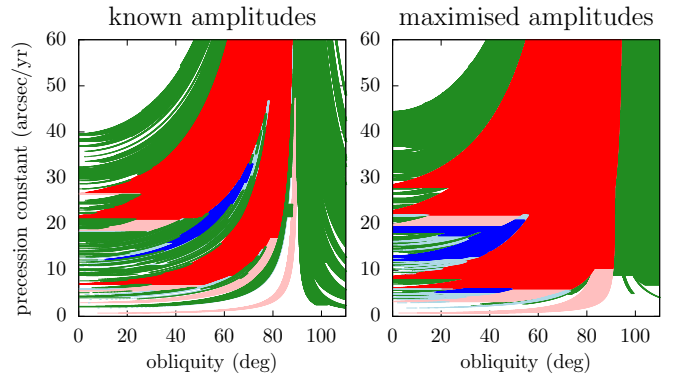


Fig. 4. Estimate of the chaotic regions of the spin-axis dynamics for the Earth, where the long-term orbital dynamics is approximated by the Lagrange-Laplace system. The same colour code as Fig. 3 is used. *Left panel:* the complete set of initial conditions is used (same as Fig. 3, bottom row). *Right panel:* the orbital elements (I, ϖ, Ω) are assumed to be unknown for all planets while the remaining ones are taken from Bretagnon (1982). Accordingly, the coefficients (E_k, S_k) of the quasi-periodic series are maximised according to the estimated AMD value (see Sect. 3.2).

constant), planet d is the most likely to suffer from large obliquity changes. One must remember, however, that the chaotic zones are maximised here according to the available orbital data. We also predict a rich obliquity dynamics for the Trappist-1 planets, but due to the confirmed strong effects of mean-motion resonances (see e.g. Quarles et al. 2017), the use of the Lagrange-Laplace model is probably inadequate in this case. Building an orbital theory specific to this system would be beyond the scope of this paper.

3.3. Maximisation of α

In Sects. 3.1 and 3.2, we saw how to obtain a quasi-periodic approximation of the long-term orbital motion of a planet, and how to estimate bounds for its coefficients if some of the orbital parameters are unknown. However, in order to study its long-term spin dynamics, we still lack an estimate of its precession constant α . Even in the solar system, the precession constants

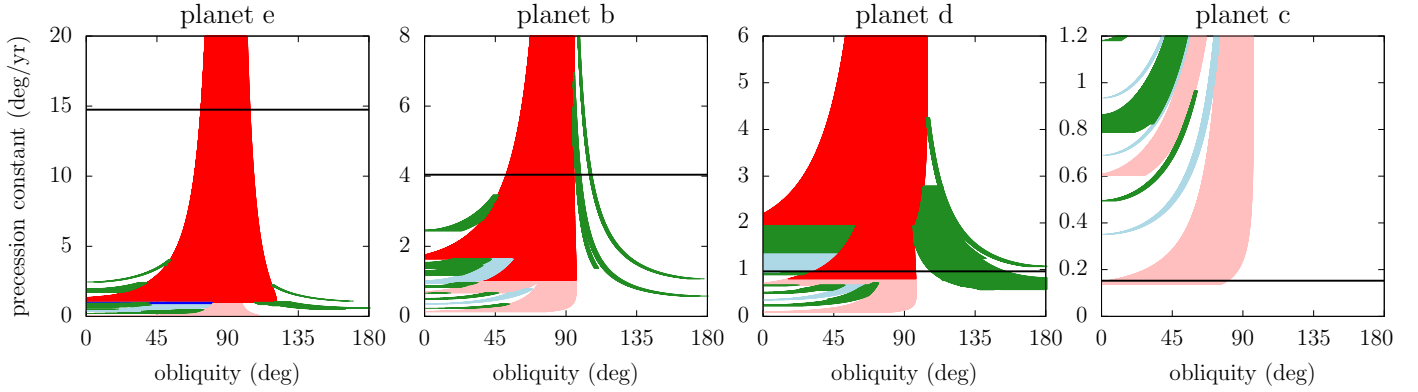


Fig. 5. Chaotic regions of the spin-axis dynamics for exoplanets of the GJ 3293 system, maximised with the method presented in this paper. The colour code is the same as in previous figures. The bounds for their precession constants are, *from left to right panels*: $\alpha_{\max} = 66, 20, 6.2$ and 1.2 deg yr^{-1} (corresponding to maximum rotation periods of a few hours). For each exoplanet, the horizontal line shows the precession constant corresponding to a rotation period equal to the orbital period (obtained from the method of Sect. 3.3). They are, from left to right: 13, 31, 48 and 123 days.

of the planets are not very well known. The estimate of α for an extrasolar planet would require observations that are very hard to obtain (its rotation period and a model of interior), and would be specific to one exoplanet. In order to keep this study as general as possible, we do not try to obtain a single value for the precession constant; instead, we look for its upper bound from general physical considerations.

After the sphere, the simplest shape model for a rotating planet is given by the Maclaurin ellipsoid (Chandrasekhar 1969). This latter describes the equilibrium shape of a self-gravitating homogeneous body in rotation with constant angular velocity. The rotational symmetry is imposed (circular equator), leading to the formula

$$\frac{\omega^2}{2\pi\mathcal{G}\rho} = \frac{\sqrt{1-\epsilon^2}}{\epsilon^3} \left((3-2\epsilon^2) \arcsin \epsilon - 3\epsilon\sqrt{1-\epsilon^2} \right), \quad (51)$$

where ω and ρ are the rotation velocity and the density of the body and ϵ is the eccentricity of its ellipsoidal figure (in any plane containing the rotation axis). Studying $f = \omega^2/(2\pi\mathcal{G}\rho)$ as a function of the ellipsoid eccentricity, f is zero for $\epsilon = 0$ and $\epsilon = 1$, and it has one maximum at $\epsilon_0 \approx 0.929956$ with the value $f_0 \approx 0.224666$. This implies that there is no such equilibrium figure possible for rotation velocities larger than

$$\omega_{\max} = \sqrt{2\pi\mathcal{G}\rho f_0}. \quad (52)$$

Converting the ellipsoid eccentricity in terms of momenta of inertia, we obtain the relation

$$\frac{2C - A - B}{2C} = \frac{1}{2}\epsilon^2. \quad (53)$$

Injecting this into the expression of the precession constant (Eq. (2)), we obtain the maximum value

$$\alpha_{\max} = \frac{3\mathcal{G}m_0}{4a^3} \frac{\epsilon_0^2}{\sqrt{2\pi\mathcal{G}\rho f_0}}. \quad (54)$$

For rotation velocities close to ω_{\max} , it is known that there exist equilibrium ellipsoidal figures with three unequal axes that have a lower total energy, called Jacobi ellipsoids (Chandrasekhar 1969). However, we only need an order of magnitude for α_{\max} and the homogeneous approximation is quite

crude anyway, allowing us to stick to Eq. (54). Planets are expected to spin much more slowly than ω_{\max} (Eq. (52)), including giant gaseous planets (Batygin 2018). In the remainder of the article, ω_{\max} is referred to as the “rotational breakup” velocity, meaning that above this limit the planet spins so fast that it experiences major structural damage.

Using the average density of the Earth, we obtain a minimum rotation period of about 2.4 h, leading to a maximum precession constant of about $230'' \text{ yr}^{-1}$. The true value for the Earth is 20 yr^{-1} , or $50'' \text{ yr}^{-1}$, if we include the additional effects of the Moon (Laskar & Robutel 1993). This remains well below our bound, but the difference with this “effective” precession constant due to the presence of satellites actually constitutes the largest source of uncertainty. Close satellites increase the effective flattening of the planet, whereas far satellites increase the effective torque from the star (Boué & Laskar 2006). In both cases, this increases the precession constant to be used in our model. This effect is particularly problematic for Saturn, because our upper bound gives $\alpha_{\max} \approx 0.75'' \text{ yr}^{-1}$, while the true value is $0.20'' \text{ yr}^{-1}$, but it increases to $0.83'' \text{ yr}^{-1}$ if we take the satellites into account (Ward & Hamilton 2004). Hence, the introduction of Saturn’s satellites makes the precession constant exceed our upper bound. This problem is unavoidable for exoplanets, because the observation of satellite systems is hard and none have been observed so far. When using our model to study the spin dynamics of exoplanets, we must therefore always keep in mind that the presence of numerous or massive satellites could modify our conclusions for borderline cases like Saturn.

Moreover, we must assume a density ρ for the exoplanets if their radius has not been measured, which adds even more uncertainty. If the radius is unknown, an order of magnitude of the density can be estimated through an empirical law adjusted to the observed mass-radius distribution (see Fig. 6). The density is, in any case, not the major source of uncertainty in our method.

3.4. Classification of non-resonant exoplanets

In Sect. 2.3, we saw that the overlap of first-order secular resonances, leading to the largest chaotic zones of the spin dynamics, can only be produced if the ratio v_i/α is smaller than 1 for at least two frequencies v_i of the inclination quasi-periodic representation. Assuming that α is bounded by α_{\max} , we can deduce that there can be no substantial chaotic zone if $v_i/\alpha_{\max} > 1$ whatever

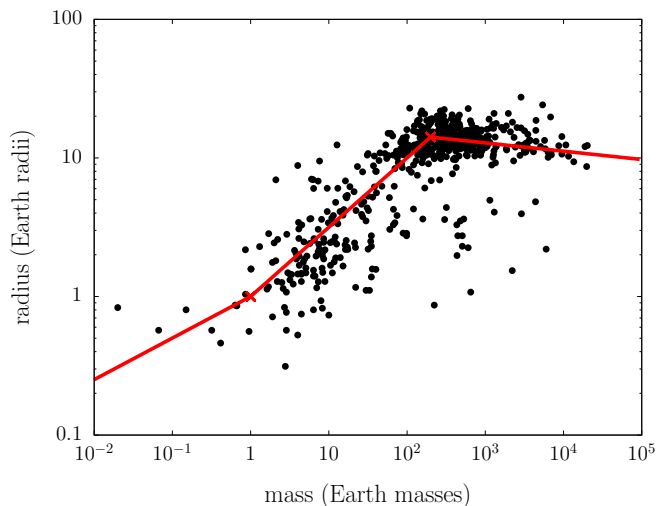


Fig. 6. Empirical mass-radius relationship obtained from the exoplanets with known mass and radius (<http://exoplanet.eu>). The exoplanets are supposed to be rocky up to 1 Earth mass, gaseous beyond 200 Earth masses, and of intermediate composition in between. Three power laws are used: $1/3$, $1/2$, and -0.06 from left to right, in general accordance with e.g. Seager et al. (2007) and Weiss et al. (2013).

the frequency ν_i . Moreover, if the mutual inclinations are small (as we assume they are), this implies that the obliquity is almost constant. Our bound for α (Eq. (54)) can thus be used for a preliminary classification of the exoplanets, while the bounds for the amplitudes (Eqs. (38) and (50)) are required for a more specific application to one exoplanet (they allow to constrain both γ and β , see Fig. 1).

Table 3 shows the 94 planets classified strictly non-resonant with this criterion, using the Lagrange-Laplace matrix to estimate the frequencies (Sect. 3.1) and Eq. (54) as a bound of α . All the exoplanets³ with known mass, semi-major axis and eccentricity were analysed (taking $m \sin I$ instead of the mass if a real-mass estimate was unavailable). On 2018-03-07, this represents 143 systems with more than one planet, which contain 353 planets in total (plus the solar system). For some of them, the frequency ratios are so far from 1 that their classification is quite safe, even when considering the numerous sources of error inherent to our method, and in particular, the possible presence of satellites. This mostly concerns planets that are far from their star, like Uranus and Neptune, and no terrestrial exoplanet has been observed yet in this category. Such large semi-major axes (third column of Table 3) imply that most of the planets listed in Table 3 are also unaffected by orbital and rotational tidal dissipation resulting from the interaction with their central star.

Most of the exoplanetary systems known so far contain only two planets. In this case, there can be no chaotic region anyway coming from the overlap of first-order resonances (Sect. 2.3) because there is only one forced frequency in inclination (Sect. 3.1). However, this single term allows us to go one step further and compute the variation range of the obliquity at first order. This is obtained by looking at the interval of parameters γ and β (Fig. 1) allowed for the exoplanet. A very simple formula can be derived if the inclination amplitude S_1 (which is the only amplitude of the decomposition) is small. Indeed, this implies that β is small as well (Eq. (26)), resulting in quite flat level curves for Colombo's top Hamiltonian (Eq. (18)). The maximum obliquity variations are achieved around $\Sigma = 0$, and at

leading order, they are equal to

$$\Delta X \approx \frac{2\beta}{\gamma} \approx 4S_1. \quad (55)$$

This approximation holds very well for small values of S_1 . For the exoplanet WASP-81 c, which has the lowest bound for S_1 in Table 3, we obtain a maximum obliquity excursion of 0.5° . This limit is very close to what is obtained by plotting the level curves of the Hamiltonian (Eq. (18)), and it holds as long as α and S_1 are below their estimated bounds. Such a good constraint cannot be achieved for every planet in two-planet systems, however, since our bound on S_1 from the AMD is sometimes not very informative. It is even dramatic for planets perturbed by a very massive companion: for example the maximum inclination amplitude of HD 92788 c is higher than 1, indicating that all inclinations are possible.

The maximum excursion of the obliquity is harder to obtain if there are more than two planets in the system, since the dynamics is ruled by the superposition of several forcing terms. However, if the maximum maximised amplitude is small (last column of Table 3), the superposition of all the terms is unlikely to bring the obliquity over the limit given at Eq. (55). It can therefore also be used as an order-of-magnitude estimate. This results in a maximum of about 20° for Uranus and Neptune, showing the crude nature of our maximisation (as shown by previous works, it is very hard to tilt Uranus and Neptune by the mean of planetary perturbations; see e.g. Boué & Laskar 2010).

4. Conclusion

The spin-axis dynamics of a planet plays a major role in its climate setting, and, by extension, in its suitability for life. However, the rotation properties of exoplanets are still very poorly constrained. In this paper, we present an analytical formulation of the long-term spin-axis dynamics of a planet, allowing to link known and unknown parameters to its obliquity evolution and to provide a global picture of the dynamics in a straightforward way.

At first, the orbital solution is modelled by quasi-periodic series. This method is therefore valid as long as the orbital chaos, if any, takes place on a much larger timescale than the spin-axis evolution. The spin-axis Hamiltonian is then expanded in powers of the eccentricity and inclination amplitudes of the orbital series. Here, we provide all terms up to order 3 but the development can be conducted to higher orders.

A clear picture of the phase space structure is given by the obliquity ranges associated with the various resonant regions. The resonant dynamics at order 1 can be characterised analytically in terms of two parameters, which are linked to the precession constant α (gathering the physical characteristics of the planet under study) and to the quasi-periodic representation of the orbit. The pendulum approximation is only used at order 2 and beyond, for which the resonances are thin enough. The regions of resonance overlap at all orders are identified as chaotic. In some cases (as for the terrestrial planets of the solar system), these chaotic regions allow wide excursions of the obliquity. The method presented here allows for the previous numerical results to be analytically retrieved with good precision. Numerical integrations prove therefore to be necessary only if detailed statistics on the obliquity evolution are required. This is very informative for solar system planets (as shown by Néron de Surgy & Laskar 1997; Correia & Laskar 2003; Laskar et al. 2004b) but not yet for exoplanets because their initial

³ <http://exoplanet.eu>

Table 3. Exoplanets from <http://exoplanet.eu> on 2018-03-07 classified non-resonant with the method detailed in this study.

Name	j/N	a (au)	α_{\max} (" yr ⁻¹)	$\frac{\min v_j }{\alpha_{\max}}$	$\max[S_i]$ ($\times 10^4$)	Name	j/N	a (au)	α_{\max} (" yr ⁻¹)	$\frac{\min v_j }{\alpha_{\max}}$	$\max[S_i]$ ($\times 10^4$)
HD 113538 b	1/2	1.24	233.0382	1.0	1815	HD 92788 c	1/2	0.60	3698.5423	8.5	10293
HD 134987 c	2/2	5.80	4.0960	1.2	763	HD 113538 c	2/2	2.44	27.9248	8.6	501
HD 163607 c	2/2	2.42	31.3391	1.2	626	HD 33844 b	1/2	1.60	190.8341	8.8	725
HD 89744 c	1/2	0.44	5497.1869	1.2	6954	Uranus	7/8	19.2	0.0682	9.9	1009
WASP-53 c	2/2	3.73	2.0657	1.3	221	HD 47366 c	2/2	1.85	131.0535	10.5	967
HD 65216 b	2/2	1.30	242.7438	1.4	596	HD 60532 b	1/2	0.77	565.4395	11.1	1291
HD 85390 c	2/2	4.23	6.5843	1.4	726	HD 73526 c	2/2	1.03	406.9428	11.1	1309
HD 183263 b	1/2	1.51	104.8361	1.5	2031	HD 128311 b	1/2	1.10	265.4641	12.2	1563
HD 204313 b	2/3	3.17	9.2689	1.5	588	HD 110014 b	2/2	2.31	28.8866	13.0	604
HD 27894 d	3/3	5.45	1.2133	1.6	268	HD 75784 c	2/2	6.50	1.2344	16.4	515
HD 204313 d	3/3	3.93	8.4220	1.7	1343	HD 89744 b	2/2	0.88	375.7932	17.2	1771
HD 37605 c	2/2	3.81	5.8515	1.8	850	HD 67087 c	2/2	3.86	6.1887	18.9	2256
HD 45364 b	1/2	0.68	1672.3835	2.0	1165	HD 142 c	2/2	6.80	0.8689	20.2	317
HD 7449 b	1/2	2.30	53.9106	2.1	6921	HD 1605 c	2/2	3.52	9.5614	20.7	202
GJ 676A b	3/4	1.81	25.8112	2.2	1731	HD 33844 c	2/2	2.24	74.3551	22.5	687
TYC+1422-614-1 b	1/2	0.69	1363.7143	2.7	630	47 UMa d	3/3	11.6	0.3274	23.4	569
HD 155358 c	2/2	1.02	647.5095	2.8	754	GJ 317 c	2/2	30.0	0.0069	23.9	2182
HD 34445 g	6/6	6.36	3.2020	2.8	162	ups And d	3/4	2.55	7.8848	23.7	768
HD 47366 b	1/2	1.21	483.1022	2.8	1270	HD 200964 b	1/2	1.60	162.1684	28.6	412
HD 73526 b	1/2	0.65	1619.2167	2.8	1647	eta Cet c	2/2	1.93	77.3953	28.9	446
HD 37124 d	3/3	2.81	30.8760	2.9	585	HD 7449 c	2/2	4.96	3.7979	29.7	2617
55 Cnc d	5/5	5.45	1.6778	3.0	34	HD 87646A c	2/2	1.58	17.3656	31.5	619
24 Sex b	1/2	1.33	287.8167	3.0	702	HD 82943 b	2/3	1.19	96.0380	32.1	1498
HD 133131A c	2/2	4.36	9.3548	3.4	2688	ups And e	4/4	5.25	5.6512	33.1	6502
HD 45364 c	2/2	0.90	965.6966	3.5	289	HD 200964 c	2/2	1.95	137.7501	33.7	771
HD 102272 c	2/2	1.57	185.6220	3.9	3092	HD 183263 c	2/2	4.25	4.5921	35.0	1163
HD 108874 c	2/2	2.68	34.1534	3.9	885	HD 82943 d	3/3	2.15	86.0321	35.8	10947
HD 147873 c	2/2	1.36	222.9728	3.9	1300	Neptune	8/8	30.1	0.0156	43.4	810
HIP 67851 c	2/2	3.82	6.7632	4.0	234	HD 5319 b	1/2	1.75	130.8026	47.8	536
HD 125612 d	3/3	4.20	3.0777	4.1	844	HD 202206 B	1/2	0.83	90.1987	49.0	1183
HD 147018 c	2/2	1.92	28.5927	4.2	357	TYC+1422-614-1 c	2/2	1.39	72.4459	51.0	110
HD 12661 c	2/2	2.56	32.4707	4.4	1213	HD 168443 c	2/2	2.84	3.9323	53.8	401
HD 74156 c	2/2	3.82	4.3343	4.7	581	HD 5319 c	2/2	2.07	107.4439	58.2	831
HD 11506 b	2/2	2.43	26.5808	5.0	462	HD 30177 b	1/2	3.58	4.4350	63.2	941
HD 141399 e	4/4	5.00	7.2669	5.0	1144	HD 128311 c	2/2	1.76	43.9586	73.5	643
HD 4732 c	2/2	4.60	7.1381	5.4	730	GJ 676 A c	4/4	6.60	0.5295	107.0	901
HD 38529 c	2/2	3.70	3.0111	5.7	86	HIP 5158 c	2/2	7.70	0.2293	134.4	168
WASP-81 c	2/2	2.43	4.6451	5.9	22	BD+202457 b	1/2	1.45	100.0644	148.1	682
HD 154857 c	2/2	5.36	4.2372	6.6	898	NN Ser (AB) d	1/2	3.39	6.7742	155.6	902
nu Oph b	1/2	1.90	45.1541	6.7	1183	HD 60532 c	2/2	1.58	39.3544	159.1	382
24 Sex c	2/2	2.08	124.2755	6.9	1300	nu Oph c	2/2	6.10	1.2729	237.0	587
Kepler-419 c	2/2	1.68	52.9030	7.0	1024	HD 30177 c	2/2	6.99	1.0682	262.5	1808
eta Cet b	1/2	1.27	317.3862	7.1	716	HD 92788 b	2/2	0.97	115.8082	272.2	266
HD 159868 b	2/2	2.25	41.0373	7.4	159	BD+202457 c	2/2	2.01	51.6913	286.7	994
HD 82943 c	1/3	0.75	391.5783	7.9	1902	HIP 57050 c	2/2	0.91	24.6886	407.6	24
mu Ara e	4/4	5.24	3.5196	8.2	405	HD 202206 c	2/2	2.41	9.7781	452.1	3521
HD 169830 c	2/2	3.60	8.7471	8.4	984	NN Ser (AB) c	2/2	5.38	0.8811	1196.6	237

Notes. The first column gives the name of the exoplanet; the second column gives the rank of the exoplanet (sorted by increasing semi-major axis) and the total number of planets in the system; the third column gives the semi-major axis value; the fourth column gives the maximum value of the precession constant estimated from Eq. (54); the fifth column gives the minimum ratio of the eigenfrequencies of the Lagrange-Laplace system and α_{\max} in absolute value (γ parameter of Colombo's top); the sixth column gives the maximum amplitude of the series decomposition obtained from the maximisation (Eq. (50)), allowing to obtain a maximum bound for the β parameter of Colombo's top.

conditions and physical parameters are still poorly known. Hence, the uncertainty of our results remains largely dominated by our lack of knowledge of the exoplanetary systems rather than by the approximations inherent to our method. This allows one to stick to the simple analytical formulas presented in this

article; no further information would be brought by numerical integrations.

At this level of uncertainty, the Lagrange-Laplace system provides a good-enough representation of the orbital motions (except for exoplanetary systems featuring highly excited orbits

or strong effects of mean-motion resonances). The formulas obtained allow for an upper bound to be set on the amplitude of the eccentricity and inclination terms if the mutual orientations of the orbits are unknown. On the other hand, the AMD equipartition hypothesis can be used, if required, to place a bound on the inclination from the eccentricity values. Through our analytical model of the spin-axis dynamics, these maximum amplitudes provide the maximum extent of the chaotic zones. For example, a large chaotic region is expected for exoplanet GJ 3293 d for rotational velocities above the synchronous rotation. Systems highly affected by mean-motion resonances (like Trappist-1) can still be studied using the method described here, but with the prior construction of a synthetic representation for the orbital motion, written in the form of a quasi-periodic series.

However, this method does not allow for tidal dissipations (playing an important role for exoplanets close to their star) to be considered, which could be modelled as an adiabatic process acting on a much longer timescale than the obliquity variations (see Néron de Surgy & Laskar 1997). Such a modelling amounts to making the precession constant α and/or the amplitudes of the orbital series gradually vary. This method does not include either the effects of libration around spin-orbit resonances, even if a method exists that allows one to take into account a possible locking in synchronous rotation (Appendix A).

Finally, under the hypothesis of hydrostatic equilibrium, we can set a bound to the precession constant α . This bound is obtained from the flattening of the planet corresponding to its rotational breakup velocity. Since α governs the width and location of the resonances, this allows one to classify the exoplanets that cannot be subject to first-order secular spin-orbit resonances. Among the sufficiently known systems with more than one planet, we found 94 planets in this category (26% of our sample). If they belong to exoplanetary systems with low mutual inclinations (as it is expected in most cases for orbital stability), this implies that their obliquity is almost constant. This bound for α is however invalidated by the possible presence of massive satellites (such as our Moon), but some exoplanets are so far from resonance that their classification is quite safe. This is the case for Uranus and Neptune.

Considering the high efficiency of the analytical method proposed here, an obliquity stability map could be designed easily in the future for each new exoplanet discovered, and in particular for those classified as “habitable”. However, such a stability map should always be computed again if any additional planet is found in the system. Indeed, it would shift the existing frequencies (especially if the new planet is massive), and add one frequency in both the inclination and eccentricity series, multiplying the possibilities of resonance. On the other hand, the total AMD of the system would increase, resulting in wider maximised chaotic zones.

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Appendix A: Case of a 1:1 spin-orbit resonance

Numerous exoplanets are observed very close to their star, in a place where the tidal frictions are strong enough to efficiently lock them in synchronous rotation. In this section, we show that if the librations around the synchronous rotation are much faster than the secular spin-axis dynamics, we can retrieve Colombo's top Hamiltonian (Sect. 2.2), allowing us to use the same approach as in the non-resonant case. As before, though, we do not consider the effect of the tidal dissipation on the obliquity. This is therefore only valid for systems for which the tidal damping of the obliquity acts on a larger timescale than the spin-axis dynamics.

We use the same method as Correia et al. (2003). We denote λ the mean longitude of the planet in orbit around the star, and ℓ its rotation angle. The mean longitude λ is measured from the equinox at a reference epoch (for instance J2000), whereas the rotation angle ℓ is measured from the equinox of the date up to a fixed point of the equator (principal axis A). If we keep the angles of the form $\ell - \lambda$ during the average over the mean longitude and the fast rotation angles (see Néron de Surgy & Laskar 1997), the corresponding "semi-averaged" Hamiltonian is

$$\begin{aligned} \mathcal{H}(L, \Lambda, Y, \ell, M, -\psi, t) = & \frac{L^2}{2C} + n\Lambda - \frac{\alpha}{2} \frac{Y^2}{L(1 - e(t)^2)^{3/2}} \\ & - \frac{\alpha_r}{2L} (L + Y)^2 \cos[2(\ell - \lambda - \psi)] \\ & - \sqrt{L^2 - Y^2} (\mathcal{A}(t) \sin \psi + \mathcal{B}(t) \cos \psi) + 2YC(t), \end{aligned} \quad (\text{A.1})$$

where we neglected terms of order $e(B - A)/C$. The momenta $L = C\omega$ and $Y = LX$ are conjugate to ℓ and $-\psi$, respectively. The momentum Λ , conjugate to λ , has been added such that $\dot{\lambda} = n$ (mean motion). The resonant precession constant is defined as

$$\alpha_r = \frac{3\mathcal{G}m_0}{8\omega a^3} \frac{B - A}{C}, \quad (\text{A.2})$$

using the same notation as Eq. (2). We note that the angle $\lambda + \psi$ appearing in the Hamiltonian corresponds to the mean longitude measured from the equinox of the date. Now we use the canonical change of coordinates

$$\begin{cases} \theta = \ell - \lambda \\ \gamma = \lambda \end{cases} \quad \text{and} \quad \begin{cases} I = L \\ \Gamma = L + \Lambda. \end{cases} \quad (\text{A.3})$$

The momentum Γ is an arbitrary constant of motion and the Hamiltonian becomes

$$\begin{aligned} \mathcal{H}(I, Y, \theta, -\psi, t) = & \frac{I^2}{2C} - nI - \frac{\alpha}{2} \frac{Y^2}{I(1 - e(t)^2)^{3/2}} \\ & - \frac{\alpha_r}{2I} (I + Y)^2 \cos(2\theta - 2\psi) \\ & - \sqrt{I^2 - Y^2} (\mathcal{A}(t) \sin \psi + \mathcal{B}(t) \cos \psi) + 2YC(t). \end{aligned} \quad (\text{A.4})$$

We now suppose that the dynamics of θ , corresponding to the "semi-secular" timescale (either circulation or oscillation), is much faster than the evolution of the other degrees of freedom, corresponding to the secular timescale. We therefore consider for now that apart from (I, θ) , all the variables are fixed (adiabatic

approximation). The equations of motion are

$$\begin{aligned} \dot{I} = -\frac{\partial \mathcal{H}}{\partial \theta} = & -\alpha_r \frac{(I + Y)^2}{I} \sin(2\theta - 2\psi) \\ \dot{\theta} = \frac{\partial \mathcal{H}}{\partial I} = & \frac{I}{C} - n + \frac{\alpha}{2} \frac{Y^2}{I^2(1 - e(t)^2)^{3/2}} \\ & - \frac{\alpha_r}{2} \frac{I^2 - Y^2}{I^2} \cos(2\theta - 2\psi) \\ & - \frac{I}{\sqrt{I^2 - Y^2}} (\mathcal{A}(t) \sin \psi + \mathcal{B}(t) \cos \psi). \end{aligned} \quad (\text{A.5})$$

Using the definitions of I , Y , and α_r , the first equation gives

$$\dot{\omega} = -\frac{3\mathcal{G}m_0}{8a^3} \frac{B - A}{C} (1 + X)^2 \sin(2\theta - 2\psi), \quad (\text{A.6})$$

resulting, for any value of X , in two equilibrium points: $\theta = \psi$ and $\psi + \pi/2 \pmod{\pi}$. We note that $\theta = \psi$ is an elliptic equilibrium while $\theta = \psi + \pi/2$ is hyperbolic. Injecting this into the second equation, we obtain

$$\dot{\theta} = \frac{I}{C} - n + \text{small terms}, \quad (\text{A.7})$$

in which the small terms correspond to the precession of the spin axis (α and α_r) and the precession of the orbit (\mathcal{A} and \mathcal{B}). The equilibrium condition, corresponding to the exact resonance, is thus $\omega \approx n$. Considering that the planet is locked in synchronous rotation, we have therefore $\theta = \psi$ and $\omega \approx n$. According to the adiabatic approximation, this will be verified whatever the value of the slow variables, such that we can inject them into the full Hamiltonian:

$$\begin{aligned} \mathcal{H}(X, -\psi, t) = & -\frac{\alpha}{2} \frac{X^2}{(1 - e(t)^2)^{3/2}} - \frac{\alpha_r}{2} (1 + X)^2 \\ & - \sqrt{1 - X^2} (\mathcal{A}(t) \sin \psi + \mathcal{B}(t) \cos \psi) + 2XC(t), \end{aligned} \quad (\text{A.8})$$

where this time, we use X as conjugate momentum of $-\psi$ (the Hamiltonian is thus divided by the constant L). In the expression of α and α_r , we must replace ω by n . We get here one extra term with respect to Eq. (1), due to the spin-orbit resonance. Using the same method as in Sect. 2.2, the Hamiltonian in case of a first-order secular spin-orbit resonance is

$$\mathcal{F}(\Sigma, \sigma) = -\frac{1}{2} (a\alpha + \alpha_r)\Sigma^2 + (b + \alpha_r)\Sigma + c \sqrt{1 - \Sigma^2} \cos \sigma, \quad (\text{A.9})$$

which must be compared to Eq. (16). This Hamiltonian has the same general form and can be reduced to Colombo's top. We can therefore apply the same method of resolution (redefining the constants accordingly).

Appendix B: Characteristic quantities of Colombo's top

B.1. Equilibrium points

From Eq. (18), the equations of motion are

$$\begin{cases} \dot{\Sigma} = -\frac{\partial \mathcal{F}}{\partial \sigma} = \beta \sqrt{1 - \Sigma^2} \sin \sigma \\ \dot{\sigma} = +\frac{\partial \mathcal{F}}{\partial \Sigma} = -\Sigma + \gamma - \beta \frac{\Sigma}{\sqrt{1 - \Sigma^2}} \cos \sigma. \end{cases} \quad (\text{B.1})$$

Apart from the coordinate singularity at $\Sigma = \pm 1$, the first equation implies that $\dot{\Sigma} = 0$ when $\sigma = 0$ or π . Injecting this into the second equation, we get

$$(\gamma - \Sigma) \sqrt{1 - \Sigma^2} = \pm \beta \Sigma \quad (\text{B.2})$$

where $\beta \geq 0$ by hypothesis. The resolution of this equation requires to square left- and right-hand terms, loosing the information⁴ about the sign of $\cos \sigma$. We obtain a quartic equation in Σ :

$$P_4(\Sigma) = \Sigma^4 - 2\gamma\Sigma^3 + (\gamma^2 + \beta^2 - 1)\Sigma^2 + 2\gamma\Sigma - \gamma^2 = 0, \quad (\text{B.3})$$

with discriminant

$$\Delta_4 = 16\gamma^2\beta^2 \left[-\gamma^6 + 3(1 - \beta^2)\gamma^4 - 3(1 + 7\beta^2 + \beta^4)\gamma^2 + (1 - \beta^2)^3 \right]. \quad (\text{B.4})$$

This discriminant is equal to zero for the particular cases $\gamma = 0$ or $\beta = 0$, for which the polynomial can be factored into, respectively,

$$P_4(\Sigma)|_{\beta=0} = (\Sigma - 1)(\Sigma + 1)(\Sigma - \gamma)^2, \quad (\text{B.5})$$

$$P_4(\Sigma)|_{\gamma=0} = (\Sigma - \sqrt{1 - \beta^2})(\Sigma + \sqrt{1 - \beta^2})\Sigma^2,$$

showing the corresponding solutions and their multiplicities. They constitute equilibrium points of the system whenever they are real and in the interval $[-1; 1]$.

For $\gamma > 0$ and $\beta > 0$, the discriminant can be either negative (two equilibrium points), zero (three equilibrium points among which one double root), or positive (four equilibrium points). The corresponding solutions can be written analytically according to the general resolution of quartic equations. They are namely

$$\begin{aligned} \Sigma_a &= \frac{1}{2}\gamma - V + \frac{1}{2}\sqrt{2C - D + \gamma\frac{1 + \beta^2}{V}}, \\ \Sigma_b &= \frac{1}{2}\gamma - V - \frac{1}{2}\sqrt{2C - D + \gamma\frac{1 + \beta^2}{V}}, \\ \Sigma_c &= \frac{1}{2}\gamma + V - \frac{1}{2}\sqrt{2C - D - \gamma\frac{1 + \beta^2}{V}}, \\ \Sigma_d &= \frac{1}{2}\gamma + V + \frac{1}{2}\sqrt{2C - D - \gamma\frac{1 + \beta^2}{V}}, \end{aligned} \quad (\text{B.6})$$

where numerous intermediary variables are required in order to obtain compact expressions:

$$\begin{aligned} W &= \gamma^2 + \beta^2 - 1, & C &= \gamma^2 - \frac{2}{3}W, \\ Z &= 108\gamma^2\beta^2 + 2W^3, & D &= \frac{1}{3}\left(U + \frac{W^2}{U}\right), \\ U &= \sqrt[3]{\frac{1}{2}\left(Z + \sqrt{Z^2 - 4W^6}\right)}, & V &= \frac{1}{2}\sqrt{C + D}. \end{aligned} \quad (\text{B.7})$$

We note that $\Sigma_{c,d}$ are real solutions only when $\Delta_4 \geq 0$ (see below for the limit in terms of γ and β). The corresponding values of σ are

$$\sigma_a = 0, \quad \sigma_b = \pi, \quad \sigma_c = \pi, \quad \sigma_d = \pi. \quad (\text{B.8})$$

The points a , b and d are elliptic fixed points, whereas the point c is hyperbolic.

⁴ After having computed one solution Σ_0 , this information is retrieved by checking the sign of $\Sigma_0/(\gamma - \Sigma_0)$.

B.2. First boundary (BC/D)

The zero value of Eq. (B.4) corresponds to a bifurcation. Its position can be computed by solving the equation $\Delta_4 = 0$, which corresponds to solving a cubic equation either in γ^2 or β^2 . Choosing to solve it in terms of β , the discriminant is

$$\Delta = -19683\gamma^4(1 + \gamma^2)^2 < 0, \quad (\text{B.9})$$

meaning that there is only one real solution. This solution is

$$\beta^2 = (1 - \gamma^{2/3})^3 \quad \text{or} \quad \gamma^2 = (1 - \beta^{2/3})^3, \quad (\text{B.10})$$

which is the boundary \mathcal{C}_1 (Eq. (20)).

B.3. Second boundary (A/B)

The other two boundaries can be obtained by studying the level curves of the Hamiltonian passing through $\Sigma = \pm 1$ (which is singular using the coordinates Σ and σ , but it does not matter here).

Let us begin with the $+1$ case, for which the Hamiltonian has a value of $-1/2 + \gamma$. We now look for this specific level curve along the axes $\sigma = 0$ and $\sigma = \pi$. This leads to the equation

$$-\frac{1}{2}\Sigma^2 + \gamma\Sigma \pm \beta\sqrt{1 - \Sigma^2} = -\frac{1}{2} + \gamma, \quad (\text{B.11})$$

for which $\Sigma = +1$ is a solution. By reorganising the terms, taking the square (thus loosing the information about the sign of $\cos \sigma$), and dividing by $(\Sigma - 1)$, we get

$$\begin{aligned} P_3(\Sigma) &= \frac{1}{4}\Sigma^3 + \left(\frac{1}{4} - \gamma\right)\Sigma^2 + \left(-\frac{1}{4} + \gamma^2 + \beta^2\right)\Sigma \\ &\quad + \left(-\frac{1}{4} + \beta^2 + \gamma - \gamma^2\right) = 0, \end{aligned} \quad (\text{B.12})$$

which is a cubic equation in Σ . Its determinant is

$$\Delta_3 = \beta^2 \left[-\beta^4 + \left(\frac{1}{4} - 5\gamma - 2\gamma^2\right)\beta^2 + \gamma(1 - \gamma)^3 \right]. \quad (\text{B.13})$$

Once again, it is zero for $\beta = 0$. Moreover the solutions for $\gamma = 0$ can be easily computed. In these two particular cases, the polynomial can be factored into, respectively,

$$\begin{aligned} P_3(\Sigma)|_{\beta=0} &= \frac{1}{4}(\Sigma - 1)(\Sigma + 1 - 2\gamma)^2, \\ P_3(\Sigma)|_{\gamma=0} &= \frac{1}{4}\left(\Sigma - \sqrt{1 - 4\beta^2}\right)\left(\Sigma + \sqrt{1 - 4\beta^2}\right)(\Sigma + 1), \end{aligned} \quad (\text{B.14})$$

showing the solutions and their multiplicities. For $\gamma > 0$ and $\beta > 0$, the discriminant can be either negative (one solution), zero (three solutions among which one double root), or positive (three solutions). The zero value corresponds to the limit we are looking for. Its position can be computed by solving the equation $\Delta_3 = 0$, which amounts to solving a quadratic equation in β^2 or a quartic equation in γ . Choosing to solve it in terms of β , the only positive solution is

$$\beta^2 = \frac{1}{8}\left(1 - 20\gamma - 8\gamma^2 + (1 + 8\gamma)^{3/2}\right), \quad (\text{B.15})$$

which is the boundary \mathcal{C}_2 (Eq. (25)).

B.4. Third boundary (B/C)

Let us now study the level curve of the Hamiltonian passing in $\Sigma = -1$, which has value $-1/2 - \gamma$. The procedure is the same as for the second boundary, and the new formulas are obtained simply by replacing γ with $-\gamma$. There is however an ambiguity because there are two positive solutions β^2 (as a function of γ) which cancel the determinant. The one corresponding to the bifurcation is the largest, that is,

$$\beta^2 = \frac{1}{8} \left(1 + 20\gamma - 8\gamma^2 + (1 - 8\gamma)^{3/2} \right), \quad (\text{B.16})$$

which is the boundary \mathcal{C}_3 (Eq. (25)).

B.5. Separatrices

The position at $\sigma = 0$ or π of the separatrix emerging from the hyperbolic point $(\Sigma, \sigma) = (\Sigma_c, \pi)$ defines the boundaries of the resonant region (see Fig. 1). Writing $f = \mathcal{F}(\Sigma_c, \pi)$, the equations to solve are

$$-\frac{1}{2}\Sigma^2 + \gamma\Sigma \pm \beta\sqrt{1 - \Sigma^2} = f. \quad (\text{B.17})$$

The resolution of this equation requires to square left- and right-hand terms, loosing the information⁵ about the sign of $\cos \sigma$. We obtain a quartic equation in Σ ,

$$\frac{1}{4}\Sigma^4 - \gamma\Sigma^3 + (\gamma^2 + \beta^2 + f)\Sigma^2 - 2f\gamma\Sigma + f^2 - \beta^2 = 0, \quad (\text{B.18})$$

in which Σ_c is a double root. It can thus be divided by $(\Sigma - \Sigma_c)^2$, leading to the quadratic equation

$$P_2(\Sigma) = \frac{1}{4}\Sigma^2 + \left(\frac{1}{2}\Sigma_c - \gamma\right)\Sigma + \left(\frac{3}{4}\Sigma_c^2 + f + \beta^2 - 2\gamma\Sigma_c + \gamma^2\right) = 0. \quad (\text{B.19})$$

This equation always has two real solutions, provided that Σ_c exists (i.e. in zones A, B, or C). These solutions are

$$\Sigma_{\pm} = 2\gamma - \Sigma_c \pm 2\sqrt{-\beta^2 + \beta\sqrt{1 - \Sigma_c^2}}, \quad (\text{B.20})$$

where we replaced f with Eq. (18) in terms of Σ_c .

Appendix C: Second-order resonances

Using the intermediary Hamiltonian $X = \varepsilon\mathcal{X}_1$ (Eq. (31)), the Hamiltonian in the new coordinates is obtained term by term from Eq. (28). The two first terms are simple: we have $\tilde{\mathcal{H}}_0 = \mathcal{H}_0$ (given at Eq. (11)) and $\tilde{\mathcal{H}}_1 = 0$ by definition of X . The second-order term is more complex since it requires computation of Poisson's brackets. Using the fact that $\{\mathcal{X}_1, \mathcal{H}_0\} = -\mathcal{H}_1$ and reorganising the terms adequately, we obtain

$$\begin{aligned} \varepsilon^2\tilde{\mathcal{H}}_2 = & -\frac{3}{4}\alpha X^2 \sum_{j=1}^N E_j^2 + 2X \sum_{j=1}^M v_j S_j^2 \\ & - 2X \sum_{j=1}^M \frac{v_j^2 S_j^2}{v_j + \alpha X} - \alpha(1 - X^2) \sum_{j=1}^M \frac{v_j^2 S_j^2}{(v_j + \alpha X)^2} \end{aligned}$$

⁵ After having computed one solution Σ_0 , this information is retrieved by checking the sign of $-\Sigma_0^2/2 + \gamma\Sigma_0 - f$.

$$\begin{aligned} & -\frac{3}{2}\alpha X^2 \sum_{j<k}^N E_j E_k \cos(\theta_j - \theta_k) \\ & + \sum_{j<k}^M S_j S_k \left[2X(v_j + v_k) - \frac{2Xv_j v_k}{v_j + \alpha X} - \frac{2Xv_j v_k}{v_k + \alpha X} \right. \\ & \quad \left. - \frac{\alpha(1 - X^2)v_j v_k}{(v_j + \alpha X)^2} - \frac{\alpha(1 - X^2)v_j v_k}{(v_k + \alpha X)^2} \right] \cos(\phi_j - \phi_k) \\ & + \alpha(1 - X^2) \sum_{j<k}^M v_j v_k S_j S_k \left[\frac{1}{(v_j + \alpha X)^2} \right. \\ & \quad \left. + \frac{1}{(v_k + \alpha X)^2} \right] \cos(\phi_j + \phi_k + 2\psi) \\ & + \alpha(1 - X^2) \sum_{j=1}^M \frac{v_j^2 S_j^2}{(v_j + \alpha X)^2} \cos(2\phi_j + 2\psi). \quad (\text{C.1}) \end{aligned}$$

Since by hypothesis there is no first-order resonance in the system, the only possible resonant angles at second order are of the form $\sigma = \phi_j + \phi_k + 2\psi$. Let us perform the canonical change of coordinates

$$\begin{pmatrix} \sigma \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\psi \\ \phi_j \\ \phi_k \end{pmatrix}, \quad (\text{C.2})$$

and

$$\begin{pmatrix} \Sigma \\ \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & 0 \\ -1 & 1 & -3 \end{pmatrix} \begin{pmatrix} X \\ \Phi_j \\ \Phi_k \end{pmatrix}. \quad (\text{C.3})$$

Assuming that σ is the only resonant angle, the dynamics at second order is given by averaging $\tilde{\mathcal{H}}$ over all other angles (this is another change of coordinates close to identity). The momenta Γ_1 and Γ_2 become arbitrary constants of motion that we conveniently choose equal to zero. Dropping the unnecessary constants, the resonant Hamiltonian is thus

$$\begin{aligned} \mathcal{F}(\Sigma, \sigma) = & -\frac{\alpha}{2}X^2 - \frac{v_j + v_k}{2}X \\ & - \frac{3}{4}\alpha X^2 \sum_{i=1}^N E_i^2 + 2X \sum_{i=1}^M v_i S_i^2 - 2X \sum_{i=1}^M \frac{v_i^2 S_i^2}{v_i + \alpha X} \\ & - \alpha(1 - X^2) \sum_{i=1}^M \frac{v_i^2 S_i^2}{(v_i + \alpha X)^2} \\ & + \alpha(1 - X^2)v_j v_k S_j S_k \left(\frac{1}{(v_j + \alpha X)^2} + \frac{1}{(v_k + \alpha X)^2} \right) \cos \sigma \end{aligned} \quad (\text{C.4})$$

in which X must be replaced by -2Σ . High-order resonances are relatively thin, so it is enough to consider the dynamics in the neighbourhood of the resonance centre at first order:

$$\begin{aligned} \dot{\sigma} = \frac{\partial \mathcal{F}}{\partial \Sigma} = & -4\alpha\Sigma + v_j + v_k + \mathcal{O}(\varepsilon^2) = 0 \\ \iff \Sigma_0 = & \frac{v_j + v_k}{4\alpha}, \end{aligned} \quad (\text{C.5})$$

or equivalently $X_0 = -2\Sigma_0 = -(v_j + v_k)/(2\alpha)$. Considering that $|X - X_0| = \mathcal{O}(\varepsilon)$, we therefore obtain

$$\mathcal{F}(\Sigma, \sigma) = -\frac{\alpha}{2}(X - X_0)^2 + \alpha K \cos \sigma, \quad (\text{C.6})$$

in which we dropped the unnecessary constants, and where

$$K = \frac{8}{(v_j - v_k)^2} \left(1 - \frac{(v_j + v_k)^2}{4\alpha^2} \right) v_j v_k S_j S_k. \quad (\text{C.7})$$

By injecting the momentum Σ instead of X and by using the modified time $d\tau = -4\alpha dt$, we obtain

$$\mathcal{F}(\Sigma, \sigma) = \frac{1}{2}(\Sigma - \Sigma_0)^2 - \frac{K}{4} \cos \sigma. \quad (\text{C.8})$$

This is the Hamiltonian of a pendulum of centre Σ_0 and half width $\sqrt{|K|}$. In terms of the obliquity cosine X , the resonance has position X_0 and half width $2\sqrt{|K|}$.

Appendix D: Third-order resonances

If there is no resonance at first and second orders, we can use a canonical change of coordinates close to identity in order to suppress the angular dependency at first and second orders. Let us consider an intermediary Hamiltonian $\mathcal{X} = \varepsilon \mathcal{X}_1 + \varepsilon^2 \mathcal{X}_2$, such that the new coordinates are given by its flow at time 1. The Hamiltonian in the new coordinates is then

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \varepsilon \tilde{\mathcal{H}}_1 + \varepsilon^2 \tilde{\mathcal{H}}_2 + \varepsilon^3 \tilde{\mathcal{H}}_3 + \mathcal{O}(\varepsilon^4), \quad (\text{D.1})$$

where

$$\begin{aligned} \tilde{\mathcal{H}}_0 &= \mathcal{H}_0, \\ \tilde{\mathcal{H}}_1 &= \mathcal{H}_1 + \{\mathcal{X}_1, \mathcal{H}_0\}, \\ \tilde{\mathcal{H}}_2 &= \mathcal{H}_2 + \{\mathcal{X}_2, \mathcal{H}_0\} + \{\mathcal{X}_1, \mathcal{H}_1\} + \frac{1}{2}\{\mathcal{X}_1, \{\mathcal{X}_1, \mathcal{H}_0\}\}, \\ \tilde{\mathcal{H}}_3 &= \mathcal{H}_3 + \{\mathcal{X}_1, \mathcal{H}_2\} + \{\mathcal{X}_2, \mathcal{H}_1\} + \frac{1}{2}\{\mathcal{X}_1, \{\mathcal{X}_2, \mathcal{H}_0\}\} \\ &\quad + \frac{1}{2}\{\mathcal{X}_2, \{\mathcal{X}_1, \mathcal{H}_0\}\} + \frac{1}{2}\{\mathcal{X}_1, \{\mathcal{X}_1, \mathcal{H}_1\}\} \\ &\quad + \frac{1}{6}\{\mathcal{X}_1, \{\mathcal{X}_1, \{\mathcal{X}_1, \mathcal{H}_0\}\}\}. \end{aligned} \quad (\text{D.2})$$

The first-order part of \mathcal{X} required to suppress the angular dependency at order 1 can be directly taken from Eq. (31). We write

$$\mathcal{A}_2 = \mathcal{H}_2 + \{\mathcal{X}_1, \mathcal{H}_1\} + \frac{1}{2}\{\mathcal{X}_1, \{\mathcal{X}_1, \mathcal{H}_0\}\} = \overline{\mathcal{A}_2} + \widetilde{\mathcal{A}_2} \quad (\text{D.3})$$

(average plus oscillating part) for which the expression is given by Eq. (C.1). The homological equation for order 2 is then

$$\{\mathcal{X}_2, \mathcal{H}_0\} + \mathcal{A}_2 = \overline{\mathcal{A}_2}, \quad (\text{D.4})$$

which defines the Hamiltonian \mathcal{X}_2 . This leads to

$$\begin{aligned} \varepsilon^2 \mathcal{X}_2 &= -\frac{3}{2} \alpha X^2 \sum_{j < k}^N \frac{E_j E_k}{\mu_j - \mu_k} \sin(\theta_j - \theta_k) \\ &\quad + \sum_{j < k}^M \frac{S_j S_k}{v_j - v_k} \left[2X(v_j + v_k) - \frac{2Xv_j v_k}{v_j + \alpha X} - \frac{2Xv_j v_k}{v_k + \alpha X} \right. \end{aligned}$$

$$\begin{aligned} &\quad \left. - \frac{\alpha(1 - X^2)v_j v_k}{(v_j + \alpha X)^2} - \frac{\alpha(1 - X^2)v_j v_k}{(v_k + \alpha X)^2} \right] \sin(\phi_j - \phi_k) \\ &\quad + \alpha(1 - X^2) \sum_{j < k}^M \frac{v_j v_k S_j S_k}{v_j + v_k + 2\alpha X} \left[\frac{1}{(v_j + \alpha X)^2} \right. \\ &\quad \quad \left. + \frac{1}{(v_k + \alpha X)^2} \right] \sin(\phi_j + \phi_k + 2\psi) \\ &\quad + \frac{1}{2} \alpha(1 - X^2) \sum_{j=1}^M \frac{v_j^2 S_j^2}{(v_j + \alpha X)^3} \sin(2\phi_j + 2\psi). \end{aligned} \quad (\text{D.5})$$

We must now compute the remainders at order 3. First of all, we can simplify their expressions by taking into account that, by definition: $\{\mathcal{X}_1, \mathcal{H}_0\} = -\mathcal{H}_1$, $\{\mathcal{X}_2, \mathcal{H}_0\} = -\widetilde{\mathcal{A}_2}$ and $\{\mathcal{X}_1, \mathcal{H}_1\} = 2(\mathcal{A}_2 - \mathcal{H}_2)$. We have then

$$\begin{aligned} \tilde{\mathcal{H}}_3 &= \mathcal{H}_3 + \frac{1}{3}\{\mathcal{X}_1, \mathcal{H}_2\} + \frac{1}{2}\{\mathcal{X}_2, \mathcal{H}_1\} \\ &\quad + \frac{1}{6}\{\mathcal{X}_1, \widetilde{\mathcal{A}_2}\} + \frac{2}{3}\{\mathcal{X}_1, \overline{\mathcal{A}_2}\}, \end{aligned} \quad (\text{D.6})$$

which gives

$$\begin{aligned} \varepsilon \tilde{\mathcal{H}}_3 &= \sum_{j=1}^M [D_j] \cos(\phi_j + \psi) \\ &\quad - \frac{3}{4} \alpha^2 (1 - X^2)^{3/2} \sum_{j=1}^M \frac{v_j^3 S_j^3}{(v_j + \alpha X)^4} \cos(3\phi_j + 3\psi) \\ &\quad + \alpha^2 (1 - X^2)^{3/2} \sum_{j=1}^M \sum_{\substack{k=1 \\ k \neq j}}^M v_j^2 v_k S_j^2 S_k [A_{jk} \\ &\quad \quad - \frac{3}{4(v_j + \alpha X)^4}] \cos(2\phi_j + \phi_k + 3\psi) \\ &\quad + \sqrt{1 - X^2} \sum_{j=1}^M \sum_{\substack{k=1 \\ k \neq j}}^M S_j^2 S_k [v_j^2 v_k B_{jk} + v_k \\ &\quad \quad + v_j \frac{(v_j + v_k)(v_k + \alpha X)}{(v_j - v_k)(v_j + \alpha X)}] \cos(2\phi_j - \phi_k + \psi) \\ &\quad + \sqrt{1 - X^2} \sum_{i=1}^M \sum_{\substack{j < k \\ j, k \neq i}}^M S_i S_j S_k [C_{ijk} \\ &\quad \quad \quad \times \cos(-\phi_i + \phi_j + \phi_k + \psi) \\ &\quad + \alpha^2 (1 - X^2)^{3/2} \sum_{i < j < k}^M v_i v_j v_k S_i S_j S_k [A_{ij} + A_{jk} \\ &\quad \quad \quad + A_{ik}] \cos(\phi_i + \phi_j + \phi_k + 3\psi) \\ &\quad + \frac{3}{2} \alpha X \sqrt{1 - X^2} \sum_{i=1}^M \sum_{j=1}^N \sum_{\substack{k=1 \\ k \neq j}}^N v_i S_i E_j E_k \left[\frac{1}{v_i + \alpha X} \right. \\ &\quad \quad \quad \left. - \frac{1}{\mu_j - \mu_k} \right] \cos(\phi_i + \theta_j - \theta_k + \psi), \end{aligned} \quad (\text{D.7})$$

where

$$x_{jk} = \frac{1}{v_j + \alpha X} + \frac{1}{v_k + \alpha X}, \quad y_{jk} = \frac{1}{(v_j + \alpha X)^2} + \frac{1}{(v_k + \alpha X)^2},$$

$$z_{jk} = \frac{1}{(v_j + \alpha X)^3} + \frac{1}{(v_k + \alpha X)^3}, \quad (\text{D.8})$$

$$b_{jk} = \frac{y_{jk}}{v_j + v_k + 2\alpha X}, \quad c_{jk} = \frac{b_{jk} + z_{jk}}{v_j + v_k + 2\alpha X},$$

$$d_{jk} = \frac{1}{3(v_j + \alpha X)(v_k + \alpha X)} \left(1 + \frac{2\alpha X}{v_j + \alpha X} - \frac{2\alpha X}{v_k + \alpha X} \right. \\ \left. + \frac{\alpha^2(1-X^2)}{(v_j + \alpha X)^2} - \frac{\alpha^2(1-X^2)}{(v_k + \alpha X)^2} + \frac{\alpha^2(1-X^2)}{(v_j + \alpha X)(v_k + \alpha X)} \right), \quad (\text{D.9})$$

$$e_{jk} = \frac{-x_{jk} + 2\alpha X y_{jk} + \alpha^2(1-X^2) z_{jk}}{v_j - v_k},$$

and

$$A_{jk} = \frac{(v_j - v_k)^2}{3(v_j + \alpha X)^3(v_k + \alpha X)^3} - c_{jk},$$

$$B_{jk} = \frac{1}{3(v_j + \alpha X)^2} \left(1 + \frac{\alpha X}{v_j + \alpha X} + \frac{5\alpha^2(1-X^2)}{4(v_j + \alpha X)^2} \right) \\ + d_{jk} + e_{jk},$$

$$C_{ijk} = 2v_i - v_j \frac{(v_i + v_k)(v_i + v_j - v_k + \alpha X)}{(v_i - v_k)(v_j + \alpha X)} \\ - v_k \frac{(v_i + v_j)(v_i + v_k - v_j + \alpha X)}{(v_i - v_j)(v_k + \alpha X)} \\ + v_i v_j v_k \left(d_{ji} + d_{ki} + 2\alpha X b_{jk} + \alpha^2(1-X^2) c_{jk} \right. \\ \left. - e_{ij} - e_{ik} + \frac{2 - 2\alpha X x_{jk} - \alpha^2(1-X^2) y_{jk}}{3(v_j + \alpha X)(v_k + \alpha X)} \right). \quad (\text{D.10})$$

The expression of the coefficients D_j is very complex. We do not give them here as they are of no interest at this stage (the angles $\phi_j + \psi$ are non-resonant by hypothesis).

As shown in Appendix C, in the pendulum approximation, the half-width of any possible resonance is two times the square root of its coefficient divided by α , and its position is given by the combination of the unperturbed frequencies. Accordingly, the possible resonances at order 3 are gathered in Table 1.

Appendix E: Geršgorin circles

In order to prove that the Lagrange-Laplace matrix for the orbital inclinations has only negative or zero eigenvalues, one can use the Geršgorin circle theorem (see Geršgorin 1931 or Varga 2004). This theorem is recalled below, and we show how it applies to our matrix.

Definition. Let B be a complex $N \times N$ matrix with elements (b_{ij}) . The i th ‘‘Geršgorin disc’’ \mathcal{G}_i ($i = 1, 2, \dots, N$) is the closed disc of the complex plane centred at b_{ii} and with radius

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^N |b_{ij}|. \quad (\text{E.1})$$

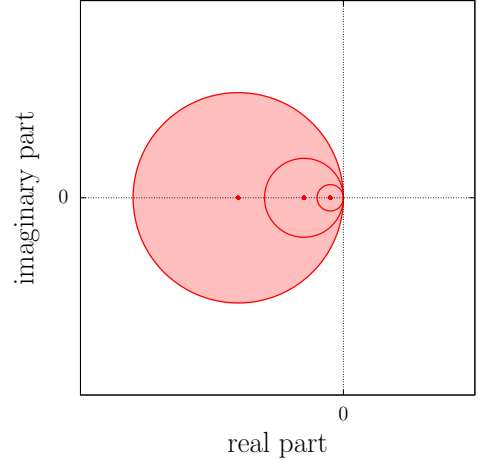


Fig. E.1. Geršgorin discs in the complex plane corresponding to the Lagrange-Laplace matrix B for three planets. The centre of the circles are the diagonal entries of B (red spots). Every eigenvalue of B lies on the real line, inside the union of all the discs.

Theorem (Geršgorin 1931). Any eigenvalue of B lies inside at least one of the \mathcal{G}_i discs, $i = 1, 2, \dots, N$.

Corollary. All the eigenvalues of B are located inside the union of the \mathcal{G}_i discs, $i = 1, 2, \dots, N$.

In our case, the matrix B is real (see Eq. (35)). It has only real eigenvalues and one of them is identically equal to zero. Moreover, given the very particular form of this matrix, the centre of each Geršgorin disc is located on the real line, with an abscissa equal to the opposite of its radius. Therefore, all the eigenvalues of B are negative or zero, as illustrated in Fig. E.1.

Appendix F: Orbital solution used for the inner solar system

In order to apply our method to a given planet, we first need a quasi-periodic approximation of its long-term orbital dynamics.

In the case of the solar system, the search for such series has been a challenge for centuries, eventually leading to very complete solutions (up to the degree of chaos inherent to the system). In the present work, we use the solution of Laskar (1990), obtained by multiplying the normalised proper modes z_i^* and ζ_i^* (Tables VI and VII of Laskar 1990) by the matrix \mathcal{S} corresponding to the linear part of the solution (Table V of Laskar 1990). In the series obtained, the terms with the same combination of frequencies are then merged together, finally resulting in 56 terms in eccentricity and 60 terms in inclination.

These series are given in Table F.1 for the inner planets, under the form:

$$z = e \exp(i\varpi) = \sum_{j=1}^N E_j \exp [i(\mu_j t + \theta_j^{(0)})], \quad (\text{F.1})$$

$$\zeta = \sin \frac{I}{2} \exp(i\Omega) = \sum_{j=1}^M S_j \exp [i(v_j t + \phi_j^{(0)})],$$

with $N = 56$ and $M = 60$. They are used in Fig. 3 of the present work.

Table F.1. Quasi-periodic representation of the orbital dynamics of Mercury.

z			ζ		
μ_j (″/yr)	$E_j \times 10^8$	$\theta_j^{(0)}$ (°)	ν_j (″/yr)	$S_j \times 10^8$	$\phi_j^{(0)}$ (°)
5.59644	18337396	110.35	-5.61755	3995819	348.70
5.47449	6902428	275.01	-7.07963	3015900	273.77
5.71670	5240271	120.52	-7.19493	1505361	105.16
4.24882	3635276	30.67	-6.96094	1429554	97.95
5.35823	2815900	94.89	-5.50098	1424811	342.89
7.45592	2786428	20.24	0.00000	1372386	107.59
4.36906	1312738	220.84	-6.84091	1183049	107.89
5.99227	1035633	113.56	-7.33264	872607	196.75
5.65485	998897	39.22	-5.85017	481844	165.47
6.93423	934569	166.16	-5.21610	360659	18.91
5.23841	829067	272.97	-5.37178	358805	35.48
7.05595	634974	357.62	-5.10025	351141	195.38
7.34103	235292	27.85	-6.73842	285961	44.50
17.91550	165568	335.25	-7.40536	264351	233.35
7.57299	164186	191.47	-7.48780	245583	47.95
17.36469	157893	303.95	-6.56016	230801	303.47
6.82468	77097	14.53	-5.96899	205822	350.64
16.81285	70120	91.98	-8.42342	192248	211.21
3.08952	60554	121.36	-3.00557	159813	140.33
18.46794	48514	9.97	-18.85115	156874	240.43
7.20563	48115	323.91	-6.15490	149031	89.77
17.08266	43934	359.38	-17.74818	119892	303.28
17.63081	39761	202.03	-0.69189	70222	23.96
7.71663	29983	273.52	18.14984	53922	111.19
28.22069	21356	307.83	-18.30007	47541	269.86
17.81084	15980	58.56	-19.40256	33322	29.01
19.01870	15738	39.75	-19.13075	14506	125.90
17.15752	15119	145.02	-26.33023	13964	127.29
18.18553	14604	57.28	-18.01114	9092	62.09
17.72293	12073	48.46	-17.66094	8620	318.93
18.01611	10275	44.83	-17.83857	7011	109.13
16.52731	9115	311.91	-17.54636	6248	66.71
17.47683	7950	80.26	-18.97001	6027	253.36
16.26122	7118	58.89	-2.35835	4687	44.73
17.55234	6398	17.65	-17.94404	4398	32.26
5.40817	6086	120.60	-18.59563	4035	278.11
18.08627	5933	356.17	-1.84625	3435	41.72
52.19257	3589	225.59	-4.16482	3303	51.62
-19.72306	2363	113.24	-18.69743	3167	41.70
4.89647	922	292.23	-18.77933	3104	42.83
0.66708	717	73.98	-18.22681	3100	226.30
1.93168	618	39.55	-19.06544	2777	230.21
3.60029	447	121.40	-17.19656	1298	127.26
-56.90922	400	44.11	-3.11725	1067	326.97
53.35188	285	134.98	-0.58033	683	17.33
29.37998	169	37.61	-1.19906	372	133.87
2.97706	158	306.81	11.50319	341	281.02
-20.88236	75	203.93	-26.97744	244	44.61
28.86795	70	212.64	-50.30212	202	29.83
27.57346	62	223.74	0.46547	196	286.88
1.82121	50	146.09	10.34389	179	191.52
27.06140	45	38.56	20.96631	132	57.78
76.16447	16	323.03	0.57829	68	103.72
0.77840	12	65.10	82.77163	61	128.95
51.03334	9	136.30	9.18847	39	1.15
-0.49216	4	164.74	58.80017	36	212.90
			34.82788	28	294.12
			-27.48935	18	218.53
			-25.17116	17	215.94
			-28.13656	11	314.08

Table F.1. continued.

z			ζ		
μ_j (″/yr)	$E_j \times 10^8$	$\theta_j^{(0)}$ (°)	ν_j (″/yr)	$S_j \times 10^8$	$\phi_j^{(0)}$ (°)
7.45592	2085594	200.24	0.00000	1377170	107.59
4.24882	1963621	30.67	-18.85115	953835	60.43
17.91550	1346128	335.25	-5.61755	671575	348.70
17.36469	1164633	123.95	-17.74818	575205	123.28
5.59644	659312	110.35	-7.07963	404368	93.77
17.08266	324058	179.38	-18.30007	298364	89.80
5.47449	248173	275.01	-5.50098	239467	342.89
16.81285	239435	274.43	-6.84091	208804	286.39
6.93423	216696	169.77	-7.19493	201837	285.16
17.63081	191660	193.67	-6.96094	191673	277.95
5.71670	188411	120.52	-19.40256	188959	209.10
7.05595	178786	358.98	-7.33264	116984	16.75
7.34103	176112	207.85	-3.00557	99208	140.33
17.81084	129713	58.56	-19.13075	88217	305.90
7.57299	122891	11.47	-5.85017	80983	165.47
18.18553	118540	57.28	-0.69189	65885	23.96
19.01870	116085	219.75	-5.21610	60616	18.91
17.15752	111520	325.02	-5.37178	60304	35.48
18.46794	110955	6.95	-5.10025	59016	195.38
5.35823	101244	94.89	-18.01114	43620	242.09
17.72293	97999	48.46	-17.66094	41358	138.93
18.01611	83405	44.83	-6.73842	38337	224.50
16.52731	67232	131.91	-18.97001	36654	73.36
17.47683	58638	260.26	-7.40536	35440	53.35
16.26122	57777	58.89	-5.96899	34592	350.64
6.82468	57706	194.53	-17.83857	33636	289.13
3.08952	54138	121.36	-7.48780	32924	227.95
18.08627	48156	356.17	-6.56016	30942	123.47
17.55234	47190	197.65	-17.54636	29977	246.71
5.99227	37236	113.56	-8.42342	25773	31.21
7.20563	36013	143.91	-6.15490	25048	89.77
5.65485	35915	39.22	-17.94404	21099	212.26
5.23841	29809	272.97	-18.59563	19361	98.11
17.71663	22441	93.52	-18.69743	19259	221.70
4.36906	20218	220.79	-18.77933	18879	222.83
28.22069	16949	308.38	-18.22681	18853	46.30
5.40817	3036	120.48	-19.06544	16888	50.21
-19.72306	1161	113.24	-17.19656	11902	171.81
0.66708	1088	73.98	18.14984	9063	111.19
27.06140	536	218.72	-26.33023	5577	127.29
4.89647	470	291.97	-2.35835	2677	44.72
29.37998	416	217.51	-1.84625	2187	40.13
52.19257	339	225.73	-4.16482	2022	51.60
28.86795	277	32.64	-3.11725	663	326.97
27.57346	244	43.74	-0.58033	641	17.33
3.60029	242	121.40	-50.30212	215	29.83
-56.90922	216	44.11	11.50319	212	281.02
2.97706	141	306.81	-1.19906	194	133.77
1.93168	69	93.94	0.46547	184	286.88
-20.88236	67	203.93	-26.97744	128	44.89
76.16447	63	143.03	10.34389	98	191.39
1.82121	46	148.00	20.96631	82	57.78
51.03334	35	316.30	0.57829	64	103.72
0.77840	19	65.10	9.18847	36	1.15
53.35188	16	135.62	82.77163	24	128.95
-0.49216	5	164.74	58.80017	14	212.90
			34.82788	11	294.12
			-27.48935	7	218.53
			-25.17116	7	215.94
			-28.13656	5	314.08

Notes. This representation is used in Fig. 3. It has been directly obtained from Laskar (1990), see text.

Table F.1. continued.

z		ζ			
μ_j ("/yr)	$E_j \times 10^8$	$\theta_j^{(0)}$ (°)	ν_j ("/yr)	$S_j \times 10^8$	$\phi_j^{(0)}$ (°)
4.24882	1891285	30.67	0.00000	1377263	107.59
7.45592	1614222	200.24	-18.85115	875509	240.43
17.91550	1315949	155.25	-5.61755	496020	348.70
17.36469	938579	303.95	-17.74818	401987	303.28
5.59644	420011	110.35	-7.07963	343071	93.77
17.08266	261159	359.38	-18.30007	281401	269.74
17.63081	197777	14.78	-5.50098	176869	342.89
28.22069	168931	128.09	-6.84091	174079	286.47
16.81285	168064	95.11	-7.19493	171242	285.16
6.93423	161978	169.87	-6.96094	162618	277.95
5.47449	158097	275.01	-19.40256	162229	29.19
7.34103	136309	207.85	-26.33023	133519	127.29
7.05595	134274	359.01	-7.33264	99249	16.75
18.46794	131495	187.69	-3.00557	89258	140.33
17.81084	126776	238.56	-19.13075	80968	125.90
5.71670	120026	120.52	-0.69189	64554	23.96
18.18553	115855	237.28	-5.85017	59814	165.47
17.72293	95780	228.46	-5.21610	44770	18.91
7.57299	95116	11.47	-5.37178	44540	35.48
19.01870	93553	39.75	-5.10025	43589	195.38
17.15752	89874	145.02	-18.97001	33642	253.36
18.01611	81516	224.83	-6.73842	32525	224.50
5.35823	64497	94.89	-18.01114	30484	62.09
3.08952	56656	121.36	-7.40536	30067	53.35
16.26122	56468	238.89	-17.66094	28903	318.93
16.52731	54183	311.91	-7.48780	27932	227.95
17.47683	47256	80.26	-6.56016	26251	123.47
18.08627	47065	176.17	-17.19656	25667	341.65
6.82468	44664	194.53	-5.96899	25550	350.64
17.55234	38030	17.65	-17.83857	23507	109.13
7.20563	27874	143.91	-8.42342	21866	31.21
5.99227	23721	113.56	-17.54636	20949	66.71
5.65485	22879	39.22	-6.15490	18500	89.77
5.23841	18989	272.97	-18.69743	17676	41.70
7.71663	17369	93.52	-18.77933	17327	42.83
4.36906	9354	220.76	-18.22681	17304	226.30
52.19257	7041	225.56	-19.06544	15500	230.21
5.40817	2871	120.45	-17.94404	14745	32.26
29.37998	1761	37.54	-18.59563	13530	278.11
27.06140	1669	38.70	18.14984	6694	111.19
-19.72306	1591	113.24	-2.35835	2098	44.69
0.66708	1259	73.98	-1.84625	1981	39.73
28.86795	1027	212.64	-4.16482	1812	51.59
27.57346	902	223.74	-26.97744	1074	43.23
53.35188	584	134.92	-0.58033	628	17.33
4.89647	447	291.91	-3.11725	596	326.97
76.16447	233	323.03	82.77163	581	128.95
3.60029	233	121.40	58.80017	341	212.90
-56.90922	208	44.11	34.82788	269	294.12
2.97706	148	306.81	11.50319	191	281.02
51.03334	129	136.30	0.46547	181	286.88
1.93168	70	148.98	-27.48935	173	218.53
-20.88236	70	203.93	-1.19906	167	133.74
1.82121	49	148.46	-25.17116	163	215.94
0.77840	22	65.10	-28.13656	108	314.08
-0.49216	6	164.74	10.34389	85	191.35
			20.96631	74	57.78
			0.57829	62	103.72
			9.18847	36	1.15
			-50.30212	25	29.78

Table F.1. continued.

z		ζ			
μ_j ("/yr)	$E_j \times 10^8$	$\theta_j^{(0)}$ (°)	ν_j ("/yr)	$S_j \times 10^8$	$\phi_j^{(0)}$ (°)
17.91550	4902750	335.25	-17.74818	3464962	303.28
17.36469	4004873	303.95	-18.85115	1541097	60.43
4.24882	2030021	30.67	0.00000	1375324	107.59
16.81285	1853846	91.90	-18.30007	745752	89.07
18.46794	1357476	9.91	-17.19656	543058	154.89
17.63081	1139332	201.65	-26.33023	457927	127.29
17.08266	1114353	359.38	-18.01114	262761	62.09
28.22069	706337	128.11	-17.66094	249135	318.93
17.81084	472390	58.56	-17.83857	202620	109.13
18.18553	431698	57.28	-17.54636	180575	66.71
19.01870	399188	39.75	-19.13075	142530	305.90
17.15752	383488	145.02	-17.94404	127098	32.26
17.72293	356895	48.46	-18.59563	116625	278.11
18.01611	303744	44.83	-5.61755	105161	348.70
7.45592	295700	200.24	-19.40256	85957	23.33
16.52731	231195	311.91	-7.07963	77142	93.77
16.26122	210411	58.89	-3.00557	63897	140.33
17.47683	201641	80.26	-0.69189	60870	23.96
18.08627	175373	356.17	-18.97001	59221	73.36
17.55234	162274	17.65	-6.84091	38863	286.50
3.08952	73627	121.36	-7.19493	38505	285.16
52.19257	26717	225.55	-5.50098	37498	342.89
6.93423	25209	170.37	-6.96094	36566	277.95
7.34103	24970	207.85	-18.69743	31116	221.70
7.05595	21406	359.16	-18.77933	30502	222.83
7.57299	17424	11.47	-18.22681	30460	46.30
6.82468	8182	194.53	-19.06544	27285	50.21
29.37998	8059	37.54	-7.33264	22307	16.75
27.06140	8027	38.70	-5.85017	12681	165.47
5.59644	6750	110.35	-5.21610	9492	18.91
7.20563	5106	143.91	-5.37178	9443	35.48
28.86795	4795	212.64	-5.10025	9241	195.38
27.57346	4212	223.74	-6.73842	7310	224.50
7.71663	3182	93.52	-7.40536	6758	53.35
4.36906	3180	40.88	-7.48780	6278	227.95
-19.72306	3055	113.24	-6.56016	5900	123.47
5.40817	2931	120.37	-5.96899	5417	350.64
5.47449	2541	275.01	-8.42342	4915	31.21
53.35188	2250	134.91	-6.15490	3922	89.77
0.66708	2008	73.98	-26.97744	3471	43.07
5.71670	1929	120.52	82.77163	1993	128.95
76.16447	1090	323.03	-1.84625	1457	38.22
5.35823	1037	94.89	18.14984	1419	111.19
51.03334	605	136.30	-4.16482	1278	51.57
4.89647	464	291.74	58.80017	1169	212.90
5.99227	381	113.56	34.82788	924	294.12
5.65485	368	39.22	-2.35835	628	44.46
5.23841	305	272.97	-0.58033	592	17.33
3.60029	250	121.40	-27.48935	592	218.53
1.93168	227	192.09	-25.17116	559	215.94
-56.90922	223	44.11	-50.30212	458	209.84
2.97706	192	306.81	-3.11725	427	326.97
-20.88236	91	203.93	-28.13656	371	314.08
1.82121	65	149.54	0.46547	170	286.88
0.77840	35	65.10	11.50319	136	281.02
-0.49216	10	164.74	-1.19906	96	133.57
			0.57829	59	103.72
			20.96631	53	57.78
			10.34389	52	191.16
			9.18847	33	1.15